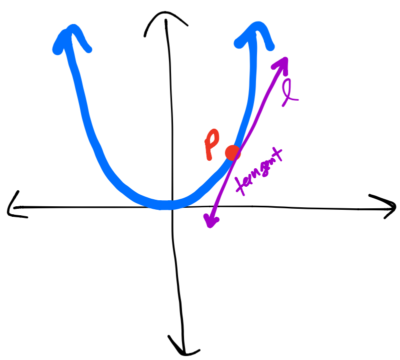
Limits & Continuity
1.5 – Slopes, Rates of Change, and Derivatives

The derivative, one of the most important concepts in all of calculus, measures the slope of a curve. This slope is the rate of change of a function. We will begin by discussing *tangent lines*.

Tangent Lines and Slopes
On a line the slope is the same at each point. On a *curve*, however, the slope may vary from point to point. To measure the slope of a curve at a specific point, we draw a *tangent line*. (The word "tangent" comes from the Latin word that means "to touch", suggesting that the tangent line just "touches" the curve.)

Tangent Lines and Slopes of Curves
The tangent line to a curve at a point P is the line through P whose steepness matches the steepness of the curve at P. The slope of this tangent line gives the slope of the curve at P.



Because the tangent line fits the curve so closely, it is called *the best linear approximation to the curve* near the point of tangency.

Explore the graph of a curve and a tangent line on your calculator

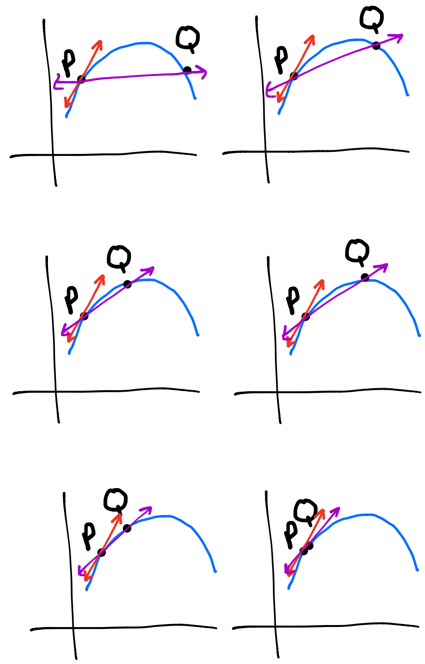
1) Graph $y = x^2$ and the tangent line to (1,1), y = 2x – 1. Use a viewing window of [-5, 5] by [-1, 6].

2) Press "Zoom", "2: Zoom In" and move the crosshairs close to the point of tangency (1,1). What do you notice?

3) Press "Zoom", "2: Zoom In" and again move the crosshairs close to the point of tangency (1,1).

What appears to be happening to the curve as you zoom in?

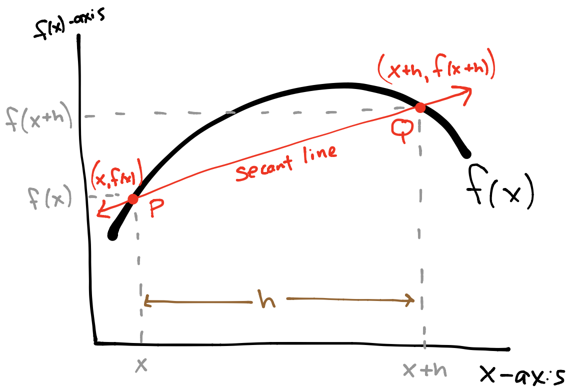
How do we define the tangent line to a curve at a point P?
A second point, Q, is chosen on the curve. Draw the *secant line* through the points P and Q. (The word "secant" comes from the Latin word that means "to cut", suggesting that the secant line "cuts" the curve at two points.)



What do you notice about the secant line and the tangent line as point Q moves closer to point P?

As Q approaches P (Q $\rightarrow$ P), the secant line becomes the tangent line at P.

Suppose a curve is defined by a function, f, that the point P has x-coordinate x, and that the point Q has x-coordinate x + h (h units farther along the x-axis).



m of secant $= \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-(x)} = \frac{f(x+h)-f(x)}{h}$

$\boxed{\therefore m \text{ of secant} = \frac{f(x+h)-f(x)}{h}}$

As noted, as $Q \to P$ the secant line becomes the tangent line . . .

$\boxed{\therefore m \text{ of tangent} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

Ex A: Find the *equation* of the tangent line to the function at the given x-value.

$f(x) = x^2$ at $x = 1$

$$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)-(x^2)}{h}$$
$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h\to 0} (2x+h)$$
$$= 2x+(0)$$
$$f'(x) = 2x \quad (m \text{ of all tangent lines})$$

Point (1,1)	Slope m=2	Point-Slope form
$f(x)=x^2$	$f'(x)=2x$	$y-y_1=m(x-x_1)$
$f(1)=(1)^2$	$f'(1)=2(1)$	$y-(1)=2(x-(1))$
$f(1)=1$	$f'(1)=2$	$y-1=2x-2$
		$y=2x-1$

The equation of the tangent line to $f(x)=x^2$ at $x=1$ is $2x-1$

The Derivative

For a function *f*, the *derivative of f at x* is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(provided that the limit exists). The derivative *f '(x)* gives the slope of the graph of *f* at *x*, and also the instantaneous rate of change of *f* at *x*.

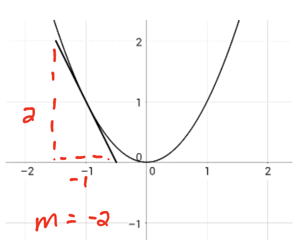
$$\frac{f(x+h) - f(x)}{h}$$

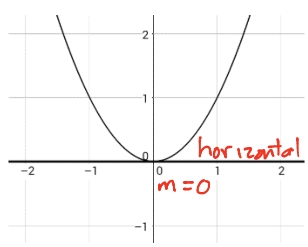
is called the *difference quotient*. It is the average slope of a line.

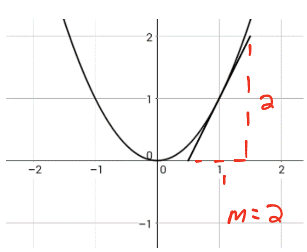
Taking $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ makes the interval shrink to an "instant," giving the *instantaneous* rate of change at *x*.

The derivative gives the slope or instantaneous rate of change at *any* value of x. For example:

At	the derivative is
x = -1	$f'(-1) =$ -2
x = 0	$f'(0) =$ 0
x = 1	$f'(1) =$ 2







Newton & Leibniz's Notation for the Derivative

Calculus was developed by two people, Isaac Newton (1642-1727), age 22, and Gottfried Wilhelm Leibniz (1646-1716), age 29. Both these individuals were from different countries; therefore, there are two different notations for the derivative.

Newton denoted derivatives by a dot over the function, $\dot{f}$. This has been replaced with the "prime" notation, *f '(x)*.

Leibniz wrote the derivative of f(x) by writing $\frac{d}{dx}$ in front of the function: $\frac{d}{dx} f(x)$. In Leibniz's notation, the fact that the derivative of x^2 is 2x is written $\frac{d}{dx} x^2 = 2x$ (*read the derivative of x^2 is 2x.)*

Prime Notation	**Leibniz's Notation**
f '(x)	$\frac{d}{dx} f(x)$
y'	$\frac{dy}{dx}$

Ex B: Finding the Derivative of a Rational Function from the Definition.

Find the derivative of $f(x) = \frac{1}{x}$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\frac{1}{x+h}\right] - \left[\frac{1}{x}\right]}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{(x+h)x} - \frac{(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - x - h}{(x+h)x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{(x+h)x}}{h}$$

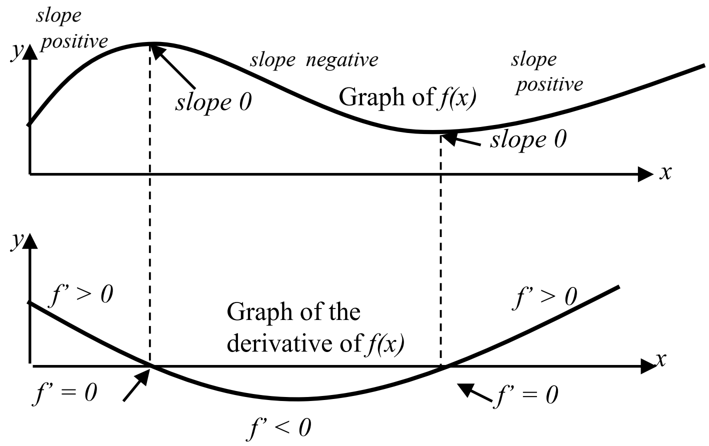
$$= \lim_{h \to 0} \frac{-h}{(x+h)x} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= \frac{-1}{[x+(0)]x}$$

$$\frac{df}{dx} = \frac{-1}{x^2}$$

Comparing the graph of a function and the graph of its derivative.



Ex C: The graph shows a function. Make a rough sketch of its derivative function.

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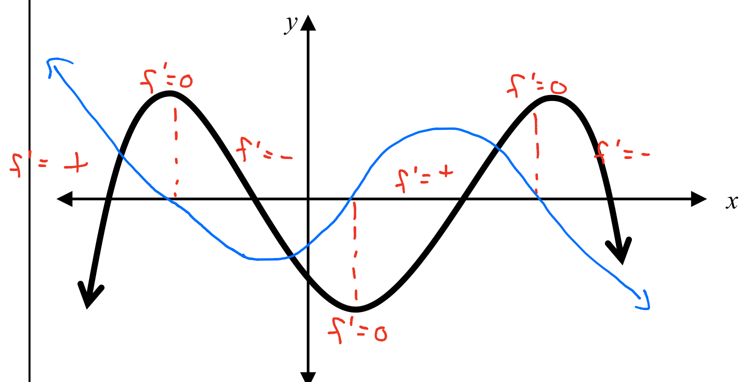

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