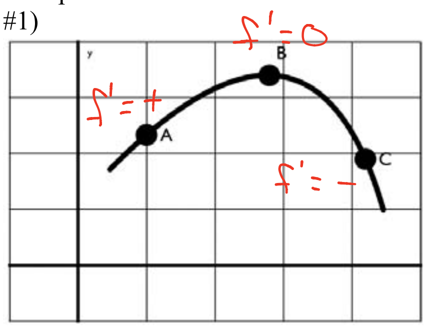
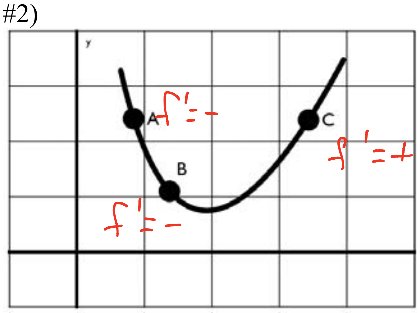
1.5A – Slopes, Rates of Change, and Derivatives

A: If a tangent line were drawn at each point, state whether the slopes are positive, negative or zero at each point

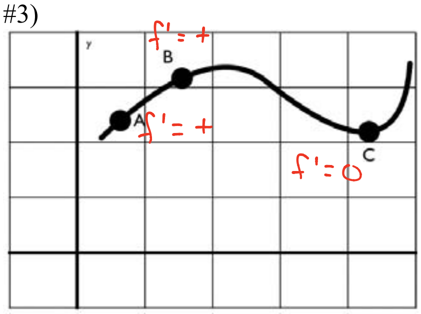
#1)



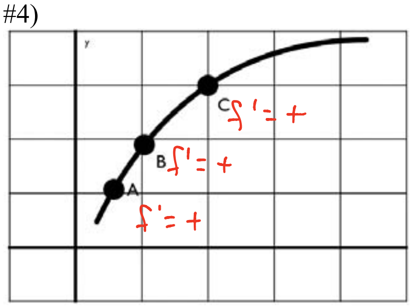
#2)



#3)

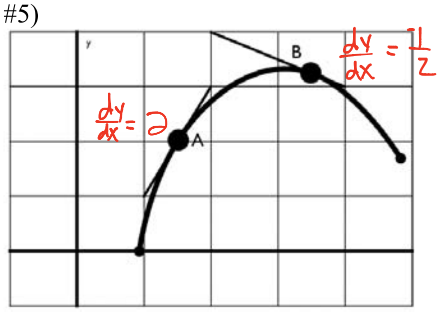


#4)

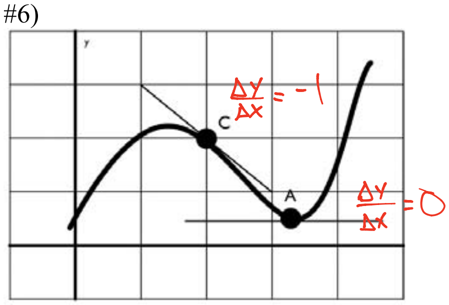


B: Find the slopes of each tangent line by counting rise and run.

#5)

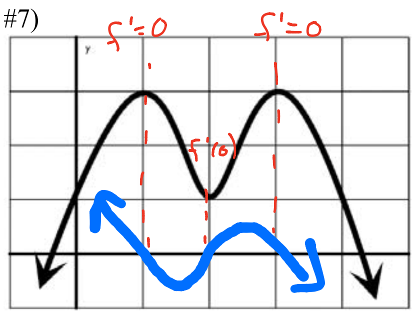


#6)

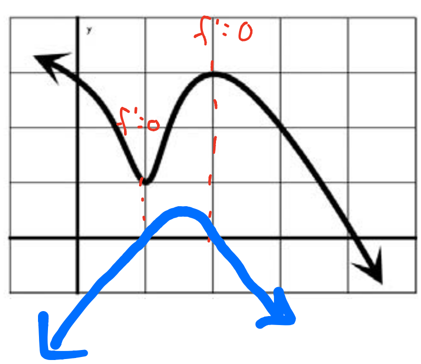


C: Use the graph of each function to make a rough sketch of the derivative showing where $f'(x)$ is positive, negative and zero.

#7)



#8)



D: Find the average rate of change (that means find the slope $m = \frac{\Delta y}{\Delta x}$) of the given function at the given x-values

#9) $y = x^2 + 7x$

a. (1, 8) and (3, 30)

$$\frac{\Delta y}{\Delta x} = \frac{(30)-(8)}{(3)-(1)} = \frac{22}{2} = 11$$

b. (1, 8) and (2, 18)

$$\frac{\Delta y}{\Delta x} = \frac{(18)-(8)}{(2)-(1)} = \frac{10}{1} = 10$$

c. (1, 8) and (1.1, 8.91)

$$\frac{\Delta y}{\Delta x} = \frac{(8.91)-(8)}{(1.1)-(1)} = \frac{.91}{.1} = 9.1$$

d. (1, 8) and (1.01, 8.0901)

$$\frac{\Delta y}{\Delta x} = \frac{(8.0901)-(8)}{(1.01)-(1)} = \frac{.0901}{.01} = 9.01$$

E: Find the instantaneous rate of change of the function, then find it specifically at the point given.

#10) $f(x) = x^2 + 7x$ at x = 1 (When finished, compare this answer and problem with #9)

$$\frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2+7(x+h)]-[x^2+7x]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2+\cancel{7x}+7h-\cancel{x^2}-\cancel{7x}}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2+7h}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h+7)}{h}$$

$$= \lim_{h\to 0} (2x+h+7)$$

$$= 2x+(0)+7$$

$$\frac{\Delta y}{\Delta x} = 2x+7$$

$$\frac{d}{dx}f(1) = 2(1)+7$$

$$= 2+7$$

$$\frac{d}{dx}f(1) = 9$$

#11) $f(x) = x^2 + 5$ at x = 3

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2+5]-[x^2+5]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2+\cancel{5}-\cancel{x^2}-\cancel{5}}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$= 2x+(0)$$

$$f'(x) = 2x$$

$$f'(3) = 2(3)$$

$$f'(3) = 6$$

#12) $f(x) = 2x - 4$ at x = 5

$$\boxed{\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[2(x+h)-4]-[2x-4]}{h} \\ &= \lim_{h\to 0} \frac{2x+2h-4-2x+4}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ f'(x) &= 2 \end{aligned}}$$

$$\boxed{f'(5) = 2}$$

#13) $f(x) = \frac{1}{x^2}$ at x = 7

$$\boxed{\begin{aligned} \frac{\Delta f}{\Delta x} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left[\frac{1}{(x+h)^2}\right]-\left[\frac{1}{x^2}\right]}{h} \\ &= \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x^2-[x^2+2hx+h^2]}{x^2(x+h)^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-2hx-h^2}{x^2(x+h)^2}}{h} \\ &= \lim_{h\to 0} \frac{h(-2x-h)}{x^2(x+h)^2}\cdot\frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\ &= \frac{-2x-(0)}{x^2[x+(0)]^2} \\ &= \frac{-2x}{x^2[x]^2} \\ &= \frac{-2x}{x^4} \\ \frac{\Delta f}{\Delta x} &= \frac{-2}{x^3} \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\Delta f(7)}{\Delta x} &= \frac{-2}{(7)^3} \\ \frac{\Delta f(7)}{\Delta x} &= \frac{-2}{343} \end{aligned}}$$

F: Find $f'(x)$ by definition.

#14) $f(x) = x^2 + x + 1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2+(x+h)+1]-[x^2+x+1]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2+\cancel{x}+h+\cancel{1}-\cancel{x^2}-\cancel{x}-\cancel{1}}{h} \\ &= \lim_{h\to 0} \frac{2hx+h^2+h}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h+1)}{h} \\ &= \lim_{h\to 0} (2x+h+1) \\ &= 2x+(0)+1 \\ f'(x) &= 2x+1 \end{aligned}$$

#15) $f(x) = 8$

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[8]-[8]}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ \frac{\Delta y}{\Delta x} &= 0 \end{aligned}$$

#16) $f(x) = \sqrt{x}$ (hint: at some point multiply by 1 in the form of $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(\sqrt{x+h})-(\sqrt{x})}{h}$$
$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h}\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x+(0)}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$

#17) $f(x) = \frac{2}{x}$

$$\frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[\frac{2}{x+h}\right]-\left[\frac{2}{x}\right]}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2x}{(x+h)x}-\frac{2(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2x-2x-2h}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-2h}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{x(x+h)}\cdot\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-2}{x(x+h)}$$
$$\frac{\Delta y}{\Delta x} = \frac{-2}{x[x+(0)]}$$
$$= \frac{-2}{x[x]}$$
$$\frac{\Delta y}{\Delta x} = \frac{-2}{x^2}$$

#18) $f(x) = \pi$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{[\pi]-[\pi]}{h}$$
$$= \lim_{h\to 0} \frac{0}{h}$$
$$= \lim_{h\to 0} 0$$
$$f'(x) = ☺$$

#19) $f(x) = \frac{x}{5}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[\frac{x+h}{5}\right]-\left[\frac{x}{5}\right]}{h}$$
$$= \lim_{h\to 0} \frac{\frac{x+h-x}{5}}{h}$$
$$= \lim_{h\to 0} \frac{h}{5}\cdot\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{1}{5}$$
$$f'(x) = \frac{1}{5}$$