Advanced Integration
10.1D – Integration by Parts

Find $\int x(4x-2)^5dx$ in two different ways.

#1) Use integration by parts.

$$\int x(4x-2)^5dx$$

$u = x$ $\quad dv = (4x-2)^5dx$

$du = dx$ $\quad v = \int p^5 \frac{dp}{4}$

$v = \frac{1}{24}p^6$

$v = \frac{1}{24}(4x-2)^6$

$p = 4x-2$

$\frac{dp}{dx} = 4$

$\frac{dp}{4} = dx$

$\int u\,dv = uv - \int v\,du$

$\int x(4x-2)^5dx = x\left(\frac{1}{24}(4x-2)^6\right) - \int \frac{1}{24}(4x-2)^6dx$

$= \frac{1}{24}x(4x-2)^6 - \frac{1}{24}\int p^6\frac{dp}{4}$

$= \frac{1}{24}x(4x-2)^6 - \frac{1}{96}\cdot\frac{1}{7}p^7 + C$

$= \frac{1}{24}x(4x-2)^6 - \frac{1}{672}(4x-2)^7 + C$

#2) Use the substitution u = 4x – 2 (so x is replaced by $\frac{1}{4}(u+2)$ and then multiply out the integrand.

$$\int x(4x-2)^5dx$$

$\int x(4x-2)^5dx$

$u+2 = 4x$

$\frac{1}{4}(u+2) = x$

$u = 4x-2$

$\frac{du}{dx} = 4$

$\frac{du}{4} = dx$

$= \int x\,u^5\frac{du}{4}$

$= \frac{1}{4}\int \frac{1}{4}(u+2)\,u^5du$

$= \frac{1}{16}\int (u^6+2u^5)\,du$

$= \frac{1}{16}\left[\frac{1}{7}u^7 + \frac{1}{3}u^6\right] + C$

$= \frac{1}{112}(4x-2)^7 + \frac{1}{48}(4x-2)^6 + C$

Derive each formula by using integration by parts on the left-handed side. (Assume n > 0).

#3) $\int x^ne^xdx = x^ne^x - n\int x^{n-1}e^xdx$

$u = x^n$ $\quad dv = e^xdx$

$du = nx^{n-1}dx$ $\quad v = e^x$

$\int u\,dv = uv - \int v\,du$

$\int x^ne^xdx = x^ne^x - \int e^x nx^{n-1}dx$

$= x^ne^x - n\int e^xx^{n-1}dx$

#4) Use the formula in number #3 to find the integral $\int x^2e^x\,dx$ (Hint: You may use a little déjà vu)

$\int x^ne^xdx = x^2e^x - 2\int x^1e^xdx$

$\int x^2e^xdx = x^2e^x - 2\int xe^xdx$

$= x^2e^x - 2\left[xe^x - 1\cdot\int x^0e^xdx\right]$

$= x^2e^x - 2\left[xe^x - e^x\right] + C$

$= x^2e^x - 2xe^x - 2e^x + C$

Advanced Integration
10.1D – Integration by Parts

I've Got Worms

#5) After watching Dumb and Dumber, George begins a new business venture called I've Got Worms. If I've Got Worms marginal revenue function is $MR(x) = xe^{x/4}$, find the revenue function.

$u = x \quad dv = e^{\frac{x}{4}}dx$

$du = dx \quad v = 4e^{\frac{x}{4}}$

$R(x) = \int xe^{\frac{x}{4}}dx$

$= uv - \int v\,du$

$= x(4e^{\frac{x}{4}}) - \int 4e^{\frac{x}{4}}dx$

$R(x) = 4xe^{\frac{x}{4}} - 16e^{\frac{x}{4}} + C$

$0 = 4(0)e^{\frac{0}{4}} - 16e^{\frac{0}{4}} + C$

$0 = 0 - 16e^{0} + C$

$0 = -16 + C$

$16 = C$

$R(x) = 4xe^{\frac{x}{4}} - 16e^{\frac{x}{4}} + 16$

Trust Fund

#6) George trusts that someone will fund his company Trust Fund Inc with a continuous stream of income of *4t* million dollars per year, where *t* is the number of years that the company has been in operation. Find the present value of this stream of income over the first 10 years at a continuous interest rate of 10%.

$PV = \int 4t\, e^{-0.10t}dt$

$u = 4t \quad dv = e^{-0.10t}dt$

$\frac{du}{dt} = 4 \quad \int dv = \int e^{-0.10t}dt$

$du = 4\,dt \quad v = -10e^{-.10t}$

$\int u\,dv = uv - \int v\,du$

$= 4t(-10e^{-.10t}) - \int -10e^{-.10t}\,4dt$

$= -40te^{-0.10t} + 40\int e^{-.10t}dt$

$= -40te^{-0.10t} - 400e^{-0.10t} + C$

$PV\Big|_0^{10} = \left[-40(10)e^{-0.10(10)} - 400e^{-0.10(10)}\right] - \left[-40(0)e^{-0.10(0)} - 400e^{-0.10(0)}\right]$

$= \left[-400e^{-1} - 400e^{-1}\right] - \left[0 - 400e^{0}\right]$

$= -\frac{800}{e} + 400$

$\simeq \$105.696447$

$\simeq \$105,696,447$

The present value is about $105.7 million

Hill Moist – Beverage of Champions

#7) George spilled his delicious sugary beverage, Hill Moist, on the floor. Being a genuine go-getter, George lofts an old unwashed, semi putrid towel on the spill. Hill Moist is absorbed into the towel at the rate of $te^{-0.5t}$ milligrams per hour, where t is the number of hours since the towel was first lofted onto the spill. Find the total amount of Hill Moist absorbed during the first 5 hours.

a. Solve $HM = \int t e^{-\frac{1}{2}t}\,dt$

$$\boxed{\begin{array}{ll} u = t & dv = e^{-\frac{1}{2}t}\,dt \\ du = dt & v = -2e^{-\frac{1}{2}t} \end{array}}$$

$$= uv - \int v\,du$$

$$= t\left(-2e^{-\frac{1}{2}t}\right) - \int -2e^{-\frac{1}{2}t}\,dt$$

$$= -2te^{-\frac{1}{2}t} - 4e^{-\frac{1}{2}t} + C$$

$$HM\Big|_0^5 = \left(-2(5)e^{-\frac{1}{2}(5)} - 4e^{-\frac{1}{2}(5)}\right) - \left(-2(0)e^{-\frac{1}{2}(0)} - 4e^{-\frac{1}{2}(0)}\right)$$

$$= \frac{-10}{e^{5/2}} - \frac{4}{e^{5/2}} + 4$$

$$= \frac{-14}{e^{5/2}} + 4$$

$$\approx 2.85 \text{ milligrams}$$

The towel will absorb about 2.85 mg during the first 5 hours.

b. Check your answer with your calculator.

Done & Done

Hill Moistened Towels

#8) George begins advertising a new product, Hill Moistened Towels, and finds that after *t* weeks the product is gaining customer recognition at the rate of $t^2 \ln t$ customers per week (for $t \geq 1$). Find the total gain in recognition from the end of week 1 to the end of week 6.

$$CR = \int t^2 \ln t\,dt$$

$$\boxed{\begin{array}{ll} u = \ln t & dv = t^2 \\ du = \frac{1}{t}dt & v = \frac{1}{3}t^3 \end{array}}$$

$$= uv - \int v\,du$$

$$= \ln t \cdot \tfrac{1}{3}t^3 - \int \tfrac{1}{3}t^3\left(\tfrac{1}{t}dt\right)$$

$$= \tfrac{1}{3}t^3 \ln t - \tfrac{1}{3}\int t^2\,dt$$

$$= \tfrac{1}{3}t^3 \ln t - \tfrac{1}{9}t^3 + C$$

$$CR\Big|_1^6 = \left(\tfrac{1}{3}(6)^3 \ln 6 - \tfrac{1}{9}(6)^3\right) - \left(\tfrac{1}{3}(1)^3 \ln 1 - \tfrac{1}{9}(1)^3\right)$$

$$= \left[\frac{216}{3}\ln 6 - \frac{216}{9}\right] - \left[\tfrac{1}{3}(1)\cdot 0 - \tfrac{1}{9}(1)\right]$$

$$= 72\ln 6 - \frac{215}{9}$$

$$\approx 105 \text{ customers}$$

From weeks 1 to 5 about 105 more customers will recognize Hill Moisened towels.

#9) Find the area under the curve $y = x \ln x$ and above the x-axis from x = 1 to x = 2.

$$A = \int_1^2 x \ln x$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\int u\,dv = uv - \int v\,du$$
$$= \ln x\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}x^2 \cdot \tfrac{1}{x}dx$$
$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx$$
$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C$$

$$A\Big|_1^2 = \left(\tfrac{1}{2}(2)^2 \ln 2 - \tfrac{1}{4}(2)^2\right) - \left(\tfrac{1}{2}(1)^2 \ln 1 - \tfrac{1}{4}(1)^2\right)$$
$$= \left[\tfrac{1}{2}(4)\ln 2 - \tfrac{1}{4}(4)\right] - \left[\tfrac{1}{2}(1)\cdot 0 - \tfrac{1}{4}(1)\right]$$
$$= \left[2\ln 2 - 1\right] - \left[-\tfrac{1}{4}\right]$$
$$= \left(2\ln 2 - \tfrac{3}{4}\right) un^2$$

Find each integral by repeating integration by parts.

#10) $\int (x+1)^2 e^x dx$

$u = (x+1)^2 \qquad dv = e^x dx$

$du = 2(x+1)(1)dx \qquad v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$\int (x+1)^2 e^x dx = (x+1)^2 e^x - \int e^x\, 2(x+1)dx$$
$$= (x+1)^2 e^x - 2\int (x+1)e^x dx$$

$u = x+1 \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= (x+1)^2 e^x - 2\left[(x+1)e^x - \int e^x dx\right]$$
$$= (x+1)^2 e^x - 2(x+1)e^x + 2e^x + C$$

#11) $\int x^2(\ln x)^2 dx$

$u = (\ln x)^2 \qquad dv = x^2 dx$

$du = 2(\ln x)\cdot\frac{1}{x}dx \qquad v = \frac{1}{3}x^3$

$\int u\,dv = uv - \int v\,du$

$\int (\ln x)^2 x^2 dx = (\ln x)^2\left(\frac{1}{3}x^3\right) - \int \frac{1}{3}x^3 \cdot 2\ln x \cdot \frac{1}{x}dx$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int \ln x \cdot x^2 dx$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\ln x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}dx\right]$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx\right]$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C\right]$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C$

#1) $\frac{1}{24}x(4x-2)^6 - \frac{1}{672}(4x-2)^7 + c$

#2) $\frac{1}{112}(4x-2)^7 + \frac{1}{48}(4x-2)^6 + c$

#3) $u = x^n$ $\quad dv = e^x dx$

#4) $x^2e^x - 2xe^x + 2e^x + c$

#5) $R(x) = 4xe^{x/4} - 16e^{\frac{x}{4}} + 16$

#6) \$105.7 million

#7) 2.85 milligrams

#8) 105 customers

#9) $\left(-\frac{3}{4} + 2\ln 2\right)un^2$

#10) $(x+1)^2e^x - 2(x+1)e^x + 2e^x + c$

#11) $\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + c$