Advanced Integration
10.1 – Integration by Parts

Basic Formulas

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq 1)$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int e^x dx = e^x + C$
4. $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$

Substitution Formulas

5. $\int u^n du = \frac{1}{n+1}u^{n+1} + C \quad (n \neq -1)$
6. $\int \frac{1}{u} du = \ln|u| + C$
7. $\int e^u du = e^u + C$

Integration by Parts Formula:

8. $\int u dv = uv - \int v\, du$ u and v are differentiable

Explanation for Formula:

For two differentiable functions u(x) and v(x), hereafter denoted *u* and *v*, the product rule is

$$(uv)' = u'v + uv'$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\begin{array}{l}\frac{dv}{dx} = v' \\ dv = v'dx\end{array}}$$

$$uv = \int v\,du + \int u\,dv$$

$$\boxed{\begin{array}{l}\frac{du}{dx} = u' \\ du = u'dx\end{array}}$$

$$uv - \int v\,du = \int u\,dv$$

Pro Tip:
Set *u* equal to expression that is easily differentiable.
Set *dv* equal to expression that is easily integrated.

Further Suggestions for Choosing *u* and *dv*.

For Integrals of the form:	Choose:
$\int x^n e^{ax} dx$	$u = x^n \quad dv = e^{ax}dx$
$\int x^n \ln x\, dx$	$u = \ln x \quad dv = x^n dx$
$\int (x+a)(x+b)^n dx$	$u = x + a \quad dv = (x+b)^n dx$

Ex. A: Integration by Parts

#1) Use integration by parts to find $\int xe^x dx$.

Derivative		Integrate
$u = x$	1	$dv = e^x dx$
$\frac{du}{dx} = 1$	2	$\int dv = \int e^x dx$
$du = dx$		$v = e^x$

3 $\int u\,dv = uv - \int v\,du$

$$\int x e^x dx = xe^x - \int e^x dx$$

4 $$= xe^x - e^x + C$$

#1: Set expressions equal to u and dv.

#2: Differentiate *u* to get *du* and integrate *dv* to get *v*.

#3: Substitute expressions for *u, du, v, and dv* into the integration by parts formula.

#4: Finally integrate the $\int v\,du$ portion of the formula.

Check the above answer by differentiating the answer. Do you get the original integral?

$$(xe^x)' - (e^x)' + (C)'$$
$$= (x)'e^x + x(e^x)' - e^x + 0$$
$$= 1\cdot e^x + xe^x - e^x$$
$$= xe^x$$

#2) $\int x^2 \ln(x)\ dx$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

$dv = x^2 dx$

$\int dv = \int x^2 dx$

$v = \frac{1}{3}x^3$

$\int u\,dv = uv - \int v\,du$

$\int \ln(x) \cdot x^2 dx = \ln(x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \left(\frac{1}{x}dx\right)$

$= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2 dx$

$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$

#3) $\int x^4 \ln(x)\ dx$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

$dv = x^4 dx$

$\int dv = \int x^4 dx$

$v = \frac{1}{5}x^5$

$\int u\,dv = uv - \int v\,du$

$\int \ln(x) \cdot x^4 dx = \ln(x) \cdot \frac{1}{5}x^5 - \int \frac{1}{5}x^5 \left(\frac{1}{x}\right)dx$

$= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\int x^4 dx$

$= \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + C$

Ex B: Which of the following integrals require *integration by parts*, and which can be found by a *u-substitution*? (Do not solve the integrals.)

#1) $\int xe^x dx$

$u = x \qquad dv = e^x dx$

Integration by parts

#2) $\int xe^{x^2} dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

u-Substitution

Ex C: Integration by parts & "p" substitution

#1) $\int (2x-1)^5(x+3)\,dx$

$u = x+3 \qquad dv = (2x-1)^5 dx$

$\frac{du}{dx} = 1 \qquad \int dv = \int (2x-1)^5 dx$

$du = dx \qquad v = \int p^5 \frac{1}{2} dp$

$v = \frac{1}{12} p^6$

$v = \frac{1}{12}(2x-1)^6$

$p = 2x-1$

$\frac{dp}{dx} = 2$

$dp = 2dx$

$\frac{1}{2}dp = dx$

$\int u\,dv = uv - \int v\,du$

$\int (x+3)(2x-1)^5 dx = (x+3)\frac{1}{12}(2x-1)^6 - \int \frac{1}{12}(2x-1)^6 dx$

$= \frac{1}{12}(x+3)(2x-1)^6 - \frac{1}{12}\int p^6 \frac{1}{2} dp$

$= \frac{1}{12}(x+3)(2x-1)^6 - \frac{1}{24}\left(\frac{1}{7}\right)p^7 + C$

$= \frac{1}{12}(x+3)(2x-1)^6 - \frac{1}{168}(2x-1)^7 + C$

#2) $\int (x-2)(3x+5)^4\,dx$

$u = x-2 \qquad dv = (3x+5)^4 dx$

$\frac{du}{dx} = 1 \qquad \int dv = \int (3x+5)^4 dx$

$du = dx \qquad v = \int p^4 \frac{dp}{3}$

$v = \frac{1}{15}p^5$

$v = \frac{1}{15}(3x+5)^5$

$p = 3x+5$

$\frac{dp}{dx} = 3$

$dp = 3dx$

$\frac{dp}{3} = dx$

$\int u\,dv = uv - \int v\,du$

$\int (x-2)(3x+5)^4 dx = (x-2)\frac{1}{15}(3x+5)^5 - \int \frac{1}{15}(3x+5)^5 dx$

$= \frac{1}{15}(x-2)(3x+5)^5 - \frac{1}{15}\int p^5 \frac{dp}{3}$

$= \frac{1}{15}(x-2)(3x+5)^5 - \frac{1}{45}\cdot\frac{1}{6}p^6 + C$

$= \frac{1}{15}(x-2)(3x+5)^5 - \frac{1}{270}(3x+5)^6 + C$

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Present Value of a Continuous Stream of Income

If a business generates income continuously at the rate C(t) dollars per year, where *t* is the number of years from now, then C(t) is called a *continuous stream of income*. Present value is the amount now that will later yield the stated sum. To find the present value of a sum under continuous compounding we multiply by e^{-rt}, where *r* is the interest rate and *t* is the number of years.

Present Value of a Continuous Stream of Income

$$Present\ Value = \int_0^T C(t)e^{-rt}\,dt$$

$C(t)$ = rate in dollars per year
t = number of years from now
T = years at continuous interest rate
r = interest rate (as a decimal)

Ex.B: Finding the Present Value of a continuous Stream of Income

President Business
#1) President Business generates income at the rate of *2t* million Lego dollars per year, where t is the number of years from now. Find the present value of this continuous stream for the next 4 years at the continuous interest rate of 10%.

$$PV = \int 2t\, e^{-0.10t}\,dt$$

$$u = 2t \qquad dv = e^{-0.10t}\,dt$$
$$\frac{du}{dt} = 2 \qquad \int dv = \int e^{-0.10t}\,dt$$
$$du = 2\,dt \qquad v = -10e^{-.10t}$$

$$\int u\,dv = uv - \int v\,du$$
$$= 2t(-10e^{-.10t}) - \int -10e^{-.10t}\,2\,dt$$
$$= -20te^{-0.10t} + 20\int e^{-.10t}\,dt$$
$$= -20te^{-0.10t} - 200e^{-0.10t} + C$$

$$\int_0^4 2t\,e^{-0.01t}\,dt = \left[-20te^{-0.10t} - 200e^{-0.10t}\right]\Big|_0^4$$
$$= \left[-20(4)e^{-0.10(4)} - 200e^{-0.10(4)}\right] - \left[-20(0)e^{-0.10(0)} - 200e^{-0.10(0)}\right]$$
$$= \left[-80e^{-0.4} - 200e^{-0.4}\right] - \left[0e^{0} - 200e^{0}\right]$$
$$= -280e^{-0.4} + 200$$
$$\approx \$12.3 \text{ billion}$$

The present value of the stream of income over 4 years is approximately \$12.3 million