Advanced Integration
10.2 – Complex Integration Formulas

Basic Formulas

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq 1)$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int e^x dx = e^x + C$
4. $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$

Substitution Formulas

5. $\int u^n du = \frac{1}{n+1}u^{n+1} + C \quad (n \neq -1)$
6. $\int \frac{1}{u} du = \ln|u| + C$
7. $\int e^u du = e^u + C$

Integration by Parts Formula

8. $\int u\, dv = uv - \int v\, du$

Complex Integration Formulas

Forms involving az + b

9. $\int \frac{z}{az+b} dz = \frac{z}{a} - \frac{b}{a^2}\ln|az+b| + C$

10. $\int \frac{1}{(az+b)(cz+d)} dz = \frac{1}{ad-bc}\ln\left|\frac{az+b}{cz+d}\right| + C$

11. $\int \frac{z}{(az+b)(cz+d)} dz = \frac{1}{ad-bc}\left(\frac{d}{c}\ln|cz+d| - \frac{b}{a}\ln|az+b|\right) + C$

12. $\int \frac{1}{z^2(az+b)} dz = -\frac{1}{b}\left(\frac{1}{z} + \frac{a}{b}\ln\left|\frac{z}{az+b}\right|\right) + C$

Forms involving $\sqrt{az+b}$

13. $\int \frac{z}{\sqrt{az+b}} dz = \frac{2az-4b}{3a^2}\sqrt{az+b} + C$

14. $\int \frac{1}{z\sqrt{az+b}} dz = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{az+b}-\sqrt{b}}{\sqrt{az+b}+\sqrt{b}}\right| + C \qquad (b > 0)$

Forms involving $z^2 - a^2$ and $a^2 - z^2$:

15. $\int \frac{1}{z^2-a^2} dz = \frac{1}{2a}\ln\left|\frac{z-a}{z+a}\right| + C$

16. $\int \frac{1}{a^2-z^2} dz = \frac{1}{2a}\ln\left|\frac{a+z}{a-z}\right| + C$

Forms involving $\sqrt{z^2 \pm a^2}$

17. $\int \sqrt{z^2 \pm a^2}\, dz = \frac{z}{2}\sqrt{z^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|z + \sqrt{z^2 \pm a^2}\right| + C$

18. $\int \frac{1}{\sqrt{z^2 \pm a^2}} dz = \ln\left|z + \sqrt{z^2 \pm a^2}\right| + C$

Forms involving $\sqrt{a^2 \pm z^2}$

19. $\int \frac{\sqrt{a^2 \pm z^2}}{z} dz = \sqrt{a^2 \pm z^2} - a\ln\left|\frac{a+\sqrt{a^2 \pm z^2}}{z}\right| + C$

20. $\int \frac{1}{z\sqrt{a^2 \pm z^2}} dz = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2 \pm z^2}}{z}\right| + C$

Reduction Formulas involving e^{az} and ln z

21. $\int z^n e^{az} dz = \frac{1}{a}z^n e^{az} - \frac{n}{a}\int z^{n-1}e^{az} dz$

22. $\int (\ln z)^n dz = z(\ln z)^n - n\int (\ln z)^{n-1} dz$

23. $\int z^n \ln z\, dz = \frac{1}{n+1}z^{n+1}\ln z - \frac{1}{(n+1)^2}z^{n+1} + C \quad (n \neq -1)$

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How To Choose a Formula

Your guess is as good as mine.

State the number of the complex integration formula you would use to find each integral. Also, state the values you would use for the substitution.

#1) $\int \frac{x}{2x+8}dx$

Formula 9: $\int \frac{z}{az+b}dz$

$z = x$ $\quad a = 2$
$dz = dx$ $\quad b = 8$

#2) $\int \frac{1}{(-3x+7)(2x-5)}dx$

Formula 10: $\int \frac{1}{(az+b)(cz+d)}dz$

$z = x$ $\quad a = -3$ $\quad c = 2$
$dz = dx$ $\quad b = 7$ $\quad d = -5$

#3) $\int \frac{x}{\sqrt{x+1}}dx$

Formula 13: $\int \frac{z}{\sqrt{az+b}}dz$

$z = x$ $\quad a = 1$
$dz = dx$ $\quad b = 1$

#4) $\int \frac{\sqrt{3-x^2}}{x}dx$

Formula 19: $\int \frac{\sqrt{a^2-z^2}}{z}dz$

$z^2 = x^2$ $\quad a^2 = 3$
$z = x$ $\quad a = \sqrt{3}$
$dz = dx$

Note: a^2 does not require the last number to be a perfect square. For example, $x^2 - 3$, can be written as $x^2 - a^2$ with $=a = \sqrt{3}$

Note : The goal is to get the left side of the formula to look like the original integral.

Integrate using one of the formulas 9 – 23.

#1) $\int \frac{1}{x^2-4}dx$

Formula 15: $\int \frac{1}{z^2-a^2}dz = \frac{1}{2a}\ln\left|\frac{z-a}{z+a}\right| + C$

$z^2 = x^2$ $\quad a^2 = 4$
$z = x$ $\quad a = 2$
$dz = dx$

$$\int \frac{1}{x^2-4}dx = \frac{1}{2(2)}\ln\left|\frac{x-2}{x+2}\right| + C$$
$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

#2) $\int \frac{x}{\sqrt{x^4+1}}dx$

Formula 18: $\int \frac{1}{\sqrt{z^2+a^2}}dz = \ln\left|z+\sqrt{z^2+a^2}\right| + C$

$z^2 = x^4$ $\quad a^2 = 1$
$z = x^2$ $\quad a = 1$
$dz = 2x\,dx$

$$\int \frac{1}{\sqrt{x^4+1}}(2x\,dx) = \ln\left|x^2+\sqrt{x^4+1}\right| + C$$
$$2\int \frac{1}{\sqrt{x^4+1}}dx =$$
$$\int \frac{1}{\sqrt{x^4+1}}dx = \frac{1}{2}\ln\left|x^2+\sqrt{x^4+1}\right| + C$$