Advanced Integration
10.3A – Complex Formulas & Substitution

Integrate using a complex formula. Some may require a manipulation of the formula.

#1) $\int x^{-99} \ln x \, dx$

Formula 23: $\int z^n \ln z \, dz = \frac{1}{n+1} z^{n+1} \ln z - \frac{1}{(n+1)^2} z^{n+1} + C$

$z = x$ $\quad n = -99$

$dz = dx$

$$\int x^{-99} \ln x \, dx = \frac{1}{-99+1} x^{-99+1} \ln x - \frac{1}{(-99+1)^2} x^{-99+1} + C$$

$$= \frac{1}{-98} x^{-98} \ln x - \frac{1}{(-98)^2} x^{-98} + C$$

#2) $\int \frac{1}{x(x+5)} dx$

Formula 10: $\int \frac{1}{(az+b)(cz+d)} dz = \frac{1}{ad-bc} \ln \left| \frac{az+b}{cz+d} \right| + C$

$z = x$ $\quad a = 1$ $\quad c = 1$

$dz = dx$ $\quad b = 0$ $\quad d = 5$

$$\int \frac{1}{(1 \cdot x + 0)(1 \cdot x + 5)} dx = \frac{1}{1 \cdot 5 - 0 \cdot 1} \ln \left| \frac{1 \cdot x + 0}{1 \cdot x + 5} \right| + C$$

$$\int \frac{1}{x(x+5)} dx = \frac{1}{5} \ln \left| \frac{x}{x+5} \right| + C$$

#3) $\int \frac{x}{x^4 - 9} dx$

Formula 15: $\int \frac{1}{z^2 - a^2} dz = \frac{1}{2a} \ln \left| \frac{z-a}{z+a} \right| + C$

$z^2 = x^4$ $\quad a^2 = 9$

$z = x^2$ $\quad a = 3$

$dz = 2x\,dx$

$$\int \frac{1}{x^4 - 9} (2x\,dx) = \frac{1}{2(3)} \ln \left| \frac{x-3}{x+3} \right| + C$$

$$2 \int \frac{x}{x^4 - 9} dx = \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| + C$$

$$\int \frac{x}{x^4 - 9} dx = \frac{1}{12} \ln \left| \frac{x-3}{x+3} \right| + C$$

#4) $\int \sqrt{4x^2 + 81} \, dx$

Formula 17: $\int \sqrt{z^2 + a^2} \, dz = \frac{z}{2} \sqrt{z^2 + a^2} + \frac{a^2}{2} \ln \left| z + \sqrt{z^2 + a^2} \right| + C$

$z^2 = 4x^2$ $\quad a^2 = 81$

$z = 2x$ $\quad a = 9$

$dz = 2dx$

$$\int \sqrt{4x^2 + 81} \; 2dx = \frac{2x}{2} \sqrt{4x^2 + 81} + \frac{81}{2} \ln \left| 2x + \sqrt{4x^2 + 81} \right| + C$$

$$2 \int \sqrt{4x^2 + 81} \, dx = x \sqrt{4x^2 + 81} + \frac{81}{2} \ln \left| 2x + \sqrt{4x^2 + 81} \right| + C$$

$$\int \sqrt{4x^2 + 81} \, dx = \frac{1}{2} x \sqrt{4x^2 + 81} + \frac{81}{4} \ln \left| 2x + \sqrt{4x^2 + 81} \right| + C$$

#5) $\int \frac{1}{\sqrt{4-e^{2t}}}dt$

Formula 20: $\int \frac{1}{z\sqrt{a^2-z^2}}dz = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-z^2}}{z}\right| + C$

$z^2 = e^{2t}$ $\quad a^2 = 4$

$z = e^t$ $\quad a = 2$

$dz = e^t dt$

$$\int \frac{1}{\cancel{e^t}\sqrt{4-e^{2t}}}\cancel{e^t}\,dt = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-e^{2t}}}{e^t}\right|$$

$$\int \frac{1}{\sqrt{4-e^{2t}}}dt = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-e^{2t}}}{e^t}\right|$$

#7) $\int \frac{x^4}{\sqrt{x^{10}-1}}dx$

Formula 18: $\int \frac{1}{\sqrt{z^2-a^2}}dz = \ln\left|z+\sqrt{z^2-a^2}\right| + C$

$z^2 = x^{10}$ $\quad a^2 = 1$

$z = x^5$ $\quad a = 1$

$dz = 5x^4 dx$

$$\int \frac{1}{\sqrt{x^{10}-1}}(5x^4 dx) = \ln\left|x^5+\sqrt{x^{10}-1}\right| + C$$

$$5\int \frac{x^4}{\sqrt{x^{10}-1}}dx = \ln\left|x^5+\sqrt{x^{10}-1}\right| + C$$

$$\int \frac{x^4}{\sqrt{x^{10}-1}}dx = \frac{1}{5}\ln\left|x^5+\sqrt{x^{10}-1}\right| + C$$

#6) $\int \frac{e^t}{e^{2t}-1}dt$

Formula 15: $\int \frac{1}{z^2-a^2}dz = \frac{1}{2a}\ln\left|\frac{z-a}{z+a}\right| + C$

$z^2 = e^{2t}$ $\quad a^2 = 1$

$z = e^t$ $\quad a = 1$

$dz = e^t dt$

$$\int \frac{1}{e^{2t}-1}e^t dt = \frac{1}{2(1)}\ln\left|\frac{e^t-1}{e^t+1}\right| + C$$

$$\int \frac{e^t}{e^{2t}-1}dt = \frac{1}{2}\ln\left|\frac{e^t-1}{e^t+1}\right| + C$$

#8) $\int \frac{1}{x\sqrt{x^3+1}}dx$

Formula 14: $\int \frac{1}{z\sqrt{az+b}}dz = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{az+b}-\sqrt{b}}{\sqrt{az+b}+\sqrt{b}}\right| + C$

$z = x^3$ $\quad a = 1$

$dz = 3x^2 dx$ $\quad b = 1$

$$\int \frac{1}{x^3\sqrt{1\cdot x^3+1}}(3x^2 dx) = \frac{1}{\sqrt{1}}\ln\left|\frac{\sqrt{1\cdot x^3+1}-\sqrt{1}}{\sqrt{1\cdot x^3+1}+\sqrt{1}}\right| + C$$

$$3\int \frac{1}{x\sqrt{x^3+1}}dx = \ln\left|\frac{\sqrt{1\cdot x^3+1}-\sqrt{1}}{\sqrt{1\cdot x^3+1}+\sqrt{1}}\right| + C$$

$$\int \frac{1}{x\sqrt{x^3+1}}dx = \frac{1}{3}\ln\left|\frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C$$