Advanced Integration
10.3B – Complex Formulas & Substitution

Find each integral by using a complex integration formula, substitution, or by parts.

#1) $\int \frac{1}{3x+6}dx$ U-subst.

$u = 3x+6$
$du = 3\,dx$
$\frac{du}{3} = dx$

$= \int (3x+6)^{-1} dx$

$= \int u^{-1}\left(\frac{du}{3}\right)$

$= \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|3x+6| + C$

#2) $\int \frac{x}{3x+6}dx$

Formula 9: $\int \frac{z}{az+b}dz = \frac{z}{a} - \frac{b}{a^2}\ln|az+b| + C$

$z = x \quad a = 3$
$dz = dx \quad b = 6$

$\int \frac{x}{3\cdot x+6}dx = \frac{x}{3} - \frac{6}{3^2}\ln|3x+6| + C$

$= \frac{x}{3} - \frac{6}{9}\ln|3x+6| + C$

$= \frac{x}{3} - \frac{2}{3}\ln|3x+6| + C$

#3) $\int x\,e^{4x}dx$ By Parts

$u = x \quad dv = e^{4x}dx$
$du = dx \quad v = \frac{1}{4}e^{4x}$

$\int u\,dv = uv - \int v\,du$

$\int xe^{4x}dx = x\cdot\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}dx$

$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$

#4) $\int x\sqrt{1-x^2}dx$ u-subst.

$u = 1-x^2$
$du = -2x\,dx$
$\frac{du}{-2x} = dx$

$= \int x\sqrt{u}\left(\frac{du}{-2x}\right)$

$= -\frac{1}{2}\int u^{\frac{1}{2}}du$

$= -\frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} + C$

$= -\frac{1}{3}\sqrt{(1-x^2)^3} + C$

#5) $\int \frac{\sqrt{1-x^2}}{x}dx$

Formula 19: $\int \frac{\sqrt{a^2-z^2}}{z}dz = \sqrt{a^2-z^2} - a\ln\left|\frac{a+\sqrt{a^2-z^2}}{z}\right| + C$

$z^2 = x^2 \quad a^2 = 1$
$z = x \quad a = 1$
$dz = dx$

$\int \frac{\sqrt{1-x^2}}{x}dx = \sqrt{1-x^2} - 1\cdot\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C$

$= \sqrt{1-x^2} - \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C$

#6) $\int \frac{x}{\sqrt{x+1}}dx$ by parts

$u = x \quad dv = (x+1)^{-\frac{1}{2}}dx$
$du = dx \quad v = \int p^{-\frac{1}{2}}dp$
$v = 2p^{\frac{1}{2}}$
$v = 2\sqrt{x+1}$

$p = x+1$
$dp = dx$

$\int u\,dv = uv - \int v\,du$

$\int x(x+1)^{-\frac{1}{2}}dx = x\cdot 2\sqrt{x+1} - \int 2\sqrt{x+1}\,dx$

$= 2x\sqrt{x+1} - 2\int (x+1)^{\frac{1}{2}}dx$

$= 2x\sqrt{x+1} - 2\int p^{\frac{1}{2}}dp$

$= 2x\sqrt{x+1} - 2\left(\frac{2}{3}\right)p^{3/2} + C$

$= 2x\sqrt{x+1} - \frac{4}{3}\sqrt{(x+1)^3} + C$

Find each integral by separating each integral into two integrals. Each integral will then require integrating using a complex integration formula.

#7) $\int \frac{x-1}{(2x+1)(x+1)}dx$ $= \int \frac{x}{(2x+1)(x+1)}dx - \int \frac{1}{(2x+1)(x+1)}dx$

$= \ln|x+1| - \frac{1}{2}\ln|2x+1| - \ln\left|\frac{2x+1}{x+1}\right| + C$

$\int \frac{x}{(2x+1)(x+1)}dx$

Formula 11: $\int \frac{z}{(az+b)(cz+d)}dz = \frac{1}{ad-bc}\left(\frac{d}{c}\ln|cz+d| - \frac{b}{a}\ln|az+b|\right) + C$

$\boxed{z=x \quad a=2 \quad c=1 \quad dz=dx \quad b=1 \quad d=1}$

$\int \frac{x}{(2\cdot x+1)(1\cdot x+1)}dx = \frac{1}{2\cdot 1-1\cdot 1}\left(\frac{1}{1}\ln|1\cdot x+1| - \frac{1}{2}\ln|2\cdot x+1|\right) + C$

$= \ln|x+1| - \frac{1}{2}\ln|2x+1| + C$

$\boxed{\frac{1}{2-1} = \frac{1}{1} = 1}$

$\int \frac{1}{(2x+1)(x+1)}dx$

Formula 10: $\int \frac{1}{(az+b)(cz+d)}dz = \frac{1}{ad-bc}\ln\left|\frac{az+b}{cz+d}\right| + C$

$\boxed{z=x \quad a=2 \quad c=1 \quad dz=dx \quad b=1 \quad d=1}$

$\boxed{\frac{1}{2\cdot 1} = \frac{1}{1} = 1}$

$\int \frac{1}{(2\cdot x+1)(1\cdot x+1)}dx = \frac{1}{2\cdot 1-1\cdot 1}\ln\left|\frac{2\cdot x+1}{1\cdot x+1}\right| + C$

$\int \frac{1}{(2x+1)(x+1)}dx = \ln\left|\frac{2x+1}{x+1}\right| + C$

#8) $\int \frac{x+1}{x\sqrt{x^2+1}}\,dx$

$$= \int \frac{x}{x\sqrt{x^2+1}}\,dx + \int \frac{1}{x\sqrt{x^2+1}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2+1}}\,dx + \int \frac{1}{x\sqrt{x^2+1}}\,dx$$

$$= \ln\left|x+\sqrt{x^2+1}\right| + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$

Formula 18: $\int \frac{1}{\sqrt{z^2+a^2}}\,dz = \ln\left|z+\sqrt{z^2+a^2}\right| + C$

$z^2 = x^2 \qquad a^2 = 1$

$z = x \qquad a = 1$

$dz = dx$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln\left|x+\sqrt{x^2+1}\right| + C$$

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx$$

Formula 14: $\int \frac{1}{z\sqrt{az+b}}\,dz = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{az+b}-\sqrt{b}}{\sqrt{az+b}+\sqrt{b}}\right| + C$

$z = x^2 \qquad a = 1$

$dz = 2x\,dx \qquad b = 1$

$$\int \frac{1}{x^2\sqrt{1\cdot x^2+1}}(2x\,dx) = \frac{1}{\sqrt{1}}\ln\left|\frac{\sqrt{1\cdot x^2+1}-\sqrt{1}}{\sqrt{1\cdot x^2+1}+\sqrt{1}}\right| + C$$

$$2\int \frac{1}{x\sqrt{x^2+1}}\,dx = \ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

Find each integral by using a reduction formula.

#9) $\int (\ln x)^2 dx$

9. $\int (\ln x)^2 dx$

Formula 22 : $\int (\ln z)^n dz = z(\ln z)^n - n\int (\ln z)^{n-1} dz$

$\boxed{z = x \quad n = 2 \quad dz = dx}$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int (\ln x)^{2-1} dx$$
$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$\boxed{n = 1}$

$$= x(\ln x)^2 - 2\left[x(\ln x)^1 - 1\int (\ln x)^{1-1} dx\right]$$
$$= x(\ln x)^2 - 2\left[x\ln x - \int (\ln x)^0 dx\right]$$
$$= x(\ln x)^2 - 2\left[x\ln x - \int 1\, dx\right]$$
$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

#10) $\int x^2 e^{3x} dx$

Formula 21: $\int z^n e^{az} = \frac{1}{a} z^n e^{az} - \frac{n}{a}\int z^{n-1} e^{az} dz$

$\boxed{z = x \quad n = 2 \quad dz = dx \quad a = 3}$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx$$

$\boxed{z = x \quad n = 1 \quad dz = dx \quad a = 3}$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int x^0 e^{3x} dx\right]$$
$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{4}{9}\int e^{3x} dx$$
$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{4}{27} e^{3x} + C$$