Advanced Integration
10.3 – Complex Reduction Formulas & Substitution

Integrate using a complex formula involving manipulation of the formula.

#1) $\int \frac{x}{\sqrt{x^4+1}}\,dx$

Formula 18: $\int \frac{1}{\sqrt{z^2+a^2}}\,dz = \ln\left|z+\sqrt{z^2+a^2}\right|+C$

$z^2 = x^4$ $\quad a^2 = 1$
$z = x^2$ $\quad a = 1$
$dz = 2x\,dx$

$\int \frac{1}{\sqrt{x^4+1}}(2x\,dx) = \ln\left|x^2+\sqrt{x^4+1}\right|+C$

$2\int \frac{x}{\sqrt{x^4+1}}\,dx =$

$\int \frac{x}{\sqrt{x^4+1}}\,dx = \frac{1}{2}\ln\left|x^2+\sqrt{x^4+1}\right|+C$

#2) $\int \frac{t}{9t^4-1}\,dt$

Formula 15: $\int \frac{1}{z^2-a^2}\,dz = \frac{1}{2a}\ln\left|\frac{z-a}{z+a}\right|+C$

$z^2 = 9t^4$ $\quad a^2 = 1$
$z = 3t^2$ $\quad a = 1$
$dz = 6t\,dt$

$\int \frac{1}{9t^4-1}(6t\,dt) = \frac{1}{2(1)}\ln\left|\frac{3t^2-1}{3t^2+1}\right|+C$

$6\int \frac{t}{9t^4-1}\,dt =$

$\int \frac{t}{9t^4-1}\,dt = \frac{1}{12}\ln\left|\frac{3t^2-1}{3t^2+1}\right|+C$

#3) $\int \frac{e^{-2t}}{e^{-t}+1}\,dt$

Formula 9: $\int \frac{z}{az+b}\,dz = \frac{z}{a}-\frac{b}{a^2}\ln|az+b|+C$

$z = e^{-t}$ $\quad a = 1$
$dz = -e^{-t}dt$ $\quad b = 1$

$\int \frac{e^{-t}}{1\cdot e^{-t}+1}(-e^{-t}dt) = \frac{e^{-t}}{1} - \frac{1}{(1)^2}\ln|1\cdot e^{-t}+1|+C$

$-\int \frac{e^{-2t}}{e^{-t}+1}\,dt =$

$\int \frac{e^{-2t}}{e^{-t}+1}\,dt = -e^{-t}+\ln|e^{-t}+1|+C$

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Find each integral by separating each integral into two integrals. Each integral will then require integrating using a complex integration formula.

#1) $\int \frac{x-1}{x^2(x+1)}dx = \int \frac{x}{x^2(x+1)}dx - \int \frac{1}{x^2(x+1)}dx$

$$= \ln\left|\frac{x}{x+1}\right| - \left(-\frac{1}{x} - \ln\left|\frac{x}{x+1}\right|\right) + C$$

$$= \ln\left|\frac{x}{x+1}\right| + \frac{1}{x} + \ln\left|\frac{x}{x+1}\right| + C$$

$$= 2\ln\left|\frac{x}{x+1}\right| + \frac{1}{x} + C$$

SIDE WORK

$\int \frac{x}{x^2(x+1)}dx = \int \frac{1}{x(x+1)}dx$

Formula 10: $\int \frac{1}{(az+b)(cz+d)}dz = \frac{1}{ad-bc}\ln\left|\frac{az+b}{cz+d}\right| + C$

$z = x$, $dz = dx$, $a = 1$, $b = 0$, $c = 1$, $d = 1$

$\int \frac{1}{(1\cdot x+0)(1\cdot x+1)}dx = \frac{1}{1\cdot 1 - 0\cdot 1}\ln\left|\frac{1\cdot x+0}{1\cdot x+1}\right| + C$

$= \frac{1}{1}\ln\left|\frac{x}{x+1}\right| + C$

$= \ln\left|\frac{x}{x+1}\right| + C$

SIDE WORK

$\int \frac{1}{x^2(x+1)}dx$

Formula 12: $\int \frac{1}{z^2(az+b)}dz = -\frac{1}{b}\left(\frac{1}{z} + \frac{a}{b}\ln\left|\frac{z}{az+b}\right|\right) + C$

$z^2 = x^2$, $z = x$, $dz = dx$, $a = 1$, $b = 1$

$\int \frac{1}{x^2(1\cdot x+1)}dx = -\frac{1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1\cdot x+1}\right|\right) + C$

$= -\frac{1}{x} - \ln\left|\frac{x}{x+1}\right| + C$

Integrate using a reduction formula.

#1) $\int x^3 e^{-x}dx$

Formula 21: $\int z^n e^{az} = \frac{1}{a}z^n e^{az} - \frac{n}{a}\int z^{n-1}e^{az}dz$

$z = x$, $dz = dx$, $n = 3$, $a = -1$

$\int x^3 e^{-x}dx = \frac{1}{-1}x^3 e^{-x} - \frac{3}{-1}\int x^2 e^{-x}dx$

$= -x^3 e^{-x} + 3\int x^2 e^{-x}dx$

$z = x$, $dz = dx$, $n = 2$, $a = -1$

$= -x^3e^{-x} + 3\left[\frac{1}{-1}x^2 e^{-x} - \frac{2}{-1}\int x e^{-x}dx\right]$

$= -x^3 e^{-x} - 3x^2 e^{-x} + 6\int x e^{-x}dx$

$z = x$, $dz = dx$, $n = 1$, $a = -1$

$= -x^3 e^{-x} - 3x^2 e^{-x} + 6\left[\frac{1}{-1}xe^{-x} - \frac{1}{-1}\int e^{-x}dx\right]$

$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + C$