Advanced Integration
10.4A – Definite Complex Integrals

Find each integral by using a complex integration formula. Give exact answers only.

#1) $\int_3^5 \sqrt{x^2-9}\,dx$

Formula 17: $\int \sqrt{z^2-a^2}\,dz = \frac{z}{2}\sqrt{z^2-a^2} - \frac{a^2}{2}\ln\left|z+\sqrt{z^2-a^2}\right| + C$

$z^2 = x^2$ $\quad a^2 = 9$

$z = x$ $\quad a = 3$

$dz = dx$

$$\int_3^5 \sqrt{x^2-9}\,dx = \left[\frac{x}{2}\sqrt{x^2-9} - \frac{9}{2}\ln\left|x+\sqrt{x^2-9}\right|\right]\Bigg|_3^5$$

$$= \left[\frac{5}{2}\sqrt{(5)^2-9} - \frac{9}{2}\ln\left|5+\sqrt{(5)^2-9}\right|\right] - \left[\frac{3}{2}\sqrt{(3)^2-9} - \frac{9}{2}\ln\left|3+\sqrt{(3)^2-9}\right|\right]$$

$$= \frac{5}{2}\sqrt{25-9} - \frac{9}{2}\ln\left|5+\sqrt{25-9}\right| - \left[\frac{3}{2}\sqrt{9-9} - \frac{9}{2}\ln\left|3+\sqrt{3-9}\right|\right]$$

$$= \frac{5}{2}\sqrt{16} - \frac{9}{2}\ln\left|5+\sqrt{16}\right| - \frac{3}{2}\sqrt{0} + \frac{9}{2}\ln\left|3+\sqrt{0}\right|$$

$$= \frac{5}{2}(4) - \frac{9}{2}\ln|5+4| - 0 + \frac{9}{2}\ln|3|$$

$$= 10 - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 3$$

$$= 10 - \frac{9}{2}\ln 3^2 + \frac{9}{2}\ln 3$$

$$= 10 - 9\ln 3 + 4.5\ln 3$$

$$= 10 - 4.5\ln 3$$

#2) $\int_2^4 \frac{1}{x^2-1}\,dx$

Formula 15: $\int \frac{1}{z^2-a^2}\,dz = \frac{1}{2a}\ln\left|\frac{z-a}{z+a}\right| + C$

$z^2 = x^2$ $\quad a^2 = 1$

$z = x$ $\quad a = 1$

$dz = dx$

$$\int_2^4 \frac{1}{x^2-1}\,dx = \left[\frac{1}{2(1)}\ln\left|\frac{x-1}{x+1}\right|\right]\Bigg|_2^4$$

$$= \left[\frac{1}{2}\ln\left|\frac{(4)-1}{(4)+1}\right|\right] - \left[\frac{1}{2}\ln\left|\frac{(2)-1}{(2)+1}\right|\right]$$

$$= \frac{1}{2}\ln\left|\frac{3}{5}\right| - \frac{1}{2}\ln\left|\frac{1}{3}\right|$$

$$= \frac{1}{2}\left[\ln 3 - \ln 5\right] - \frac{1}{2}\left[\ln 1 - \ln 3\right]$$

$$= \frac{1}{2}\left[\ln 3 - \ln 5 - 0 + \ln 3\right]$$

$$= \frac{1}{2}\left[2\ln 3 - \ln 5\right]$$

$$= \ln 3 - \frac{1}{2}\ln 5$$

$$= \ln 3 - \ln\sqrt{5}$$

#3) $\int_3^5 \frac{\sqrt{25-x^2}}{x}\,dx$

Formula 19: $\int \frac{\sqrt{a^2-z^2}}{z}\,dz = \sqrt{a^2-z^2} - a\ln\left|\frac{a+\sqrt{a^2-z^2}}{z}\right| + C$

$\boxed{\begin{array}{ll} z^2=x^2 & a^2=25 \\ z=x & a=5 \\ dz=dx & \end{array}}$

$\int_3^5 \frac{\sqrt{25-x^2}}{x}\,dx = \left[\sqrt{25-x^2} - 5\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right|\right]\Bigg|_3^5$

$= \left[\sqrt{25-(5)^2} - 5\ln\left|\frac{5+25\sqrt{25-(5)^2}}{5}\right|\right]$

$- \left[\sqrt{25-(3)^2} - 5\ln\left|\frac{5+\sqrt{25-(3)^2}}{3}\right|\right]$

$= \sqrt{25-25} - 5\ln\left|\frac{5+25\sqrt{25-25}}{5}\right|$

$- \left[\sqrt{25-9} - 5\ln\left|\frac{5+\sqrt{25-9}}{3}\right|\right]$

$= \sqrt{0} - 5\ln\left|\frac{5+25\sqrt{0}}{5}\right| - \sqrt{16} + 5\ln\left|\frac{5+\sqrt{16}}{3}\right|$

$= 0 - 5\ln\left|\frac{5}{5}\right| - 4 + 5\ln\left|\frac{5+4}{3}\right|$

$= -5\ln 1 - 4 + 5\ln\left|\frac{9}{3}\right|$

$= -5(0) - 4 + 5\ln 3$

$= -4 + 5\ln 3$

#4) $\int_4^5 \frac{1}{x\sqrt{25-x^2}}\,dx$

Formula 20: $\int \frac{1}{z\sqrt{a^2-z^2}}\,dz = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-z^2}}{z}\right| + C$

$\boxed{\begin{array}{ll} z^2=x^2 & a^2=25 \\ z=x & a=5 \\ dz=dx & \end{array}}$

$\int_4^5 \frac{1}{x\sqrt{25-x^2}}\,dx = \left[-\frac{1}{5}\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right|\right]\Bigg|_4^5$

$= \left[-\frac{1}{5}\ln\left|\frac{5+\sqrt{25-(5)^2}}{5}\right|\right] - \left[-\frac{1}{5}\ln\left|\frac{5+\sqrt{25-(4)^2}}{4}\right|\right]$

$= -\frac{1}{5}\ln\left|\frac{5+\sqrt{25-25}}{5}\right| + \frac{1}{5}\ln\left|\frac{5+\sqrt{25-16}}{4}\right|$

$= -\frac{1}{5}\ln\left|\frac{5+\sqrt{0}}{5}\right| + \frac{1}{5}\ln\left|\frac{5+\sqrt{9}}{4}\right|$

$= -\frac{1}{5}\ln\left|\frac{5}{5}\right| + \frac{1}{5}\ln\left|\frac{5+3}{4}\right|$

$= -\frac{1}{5}\ln 1 + \frac{1}{5}\ln\frac{8}{4}$

$= -\frac{1}{5}(0) + \frac{1}{5}\ln 2$

$= \frac{1}{5}\ln 2$

Anti-George Gene

#5) Under laboratory trials, the number of generations necessary to increase the frequency of the anti-George gene from 0.1 to 0.3 is

$$n = \int_{0.1}^{0.3} \frac{1}{x^2(-x+1)}dx$$

Find n to the nearest integer.

Formula 12: $\int \frac{1}{z^2(az+b)}dz = -\frac{1}{b}\left(\frac{1}{z}+\frac{a}{b}\ln\left|\frac{z}{az+b}\right|\right)+C$

$\boxed{\begin{array}{ll} z^2=x^2 & a=-1 \\ z=x & b=1 \\ dz=dx & \end{array}}$

$$n = \int_{0.1}^{0.3}\frac{1}{x^2(-1\cdot x+1)}dx = -\frac{1}{1}\left(\frac{1}{x}+\frac{-1}{1}\ln\left|\frac{x}{-1\cdot x+1}\right|\right)\Bigg|_{0.1}^{0.3}$$

$$= \left[-\frac{1}{x}+\ln\left|\frac{x}{-x+1}\right|\right]\Bigg|_{0.1}^{0.3}$$

$$= \left[\frac{1}{-0.3}+\ln\left|\frac{0.3}{-0.3+1}\right|\right]-\left[-\frac{1}{0.1}+\ln\left|\frac{0.1}{-0.1+1}\right|\right]$$

$$= -\frac{10}{3}+\ln\left|\frac{0.3}{0.7}\right|+10-\ln\left|\frac{0.1}{0.9}\right|$$

$$= -\frac{10}{3}+\frac{30}{3}+\ln\left|\frac{3}{7}\right|-\ln\left|\frac{1}{9}\right|$$

$$= \frac{20}{3}+\ln\frac{3}{7}-\ln\frac{1}{9}$$

$\simeq 8$ generations

It will take 8 generations to increase anti-George gene from 0.1 to 0.3

Ant Hill

#6) The population of George's ant hill is expected to grow at a rate of $\frac{x}{\sqrt{x+16}}$ thousand ants per month after x months. Find the total change in ants from month 9 to month 20.

$$\left(\begin{array}{c}\text{week 9 to 20} \\ \text{total}\end{array}\right) = \int_9^{20}\frac{x}{\sqrt{x+16}}dx$$

Formula 13: $\int\frac{z}{\sqrt{az+b}}dz = \frac{2az-4b}{3a^2}\sqrt{az+b}+C$

$\boxed{\begin{array}{ll} z=x & a=1 \\ dz=dx & b=16 \end{array}}$

$$\int_9^{20}\frac{x}{\sqrt{1\cdot x+9}}dx = \left[\frac{2(1)x-4(16)}{3(1)^2}\sqrt{1\cdot x+16}\right]\Bigg|_9^{20}$$

$$= \left[\frac{2(1)(20)-4(16)}{3}\sqrt{1(20)+16}\right]-\left[\frac{2(1)(9)-4(16)}{3}\sqrt{1(9)+16}\right]$$

$$= \left[\frac{40-64}{3}\sqrt{20+16}\right]-\left[\frac{18-64}{3}\sqrt{9+16}\right]$$

$$= \frac{-24}{3}\sqrt{36}-\frac{-46}{3}\sqrt{25}$$

$$= \frac{-24}{\cancel{3}}(\cancel{6}^{2})+\frac{46}{3}(5)$$

$$= -48+\frac{230}{3}$$

$\simeq 28.667$ thousand ants.

About 28,667 ants were added to the population from month 9 to month 20.

Computer Chips
George decides to capitalize on America's insatiable desire to eat potato chips. He starts manufacturing his potato chips in his factory, and names his company Computer Chips. The marginal cost function for a bag of Computer Chip is $MC(x) = \int \frac{1}{\sqrt{x^2+2}}dx$ and fixed costs are $2000.

#7) Find the cost function for bags of Computer Chips.

Formula 18: $\int \frac{1}{\sqrt{z^2+a^2}}dz = \ln\left|z+\sqrt{z^2+a^2}\right| + C$

$$\boxed{\begin{array}{ll} z^2 = x^2 & a^2 = 2 \\ z = x & a = \sqrt{2} \\ dz = dx & \end{array}}$$

$C(x) = \int \frac{1}{\sqrt{x^2+2}}dx = \ln\left|x+\sqrt{x^2+2}\right| + C$

$C(x)\big|_{(0,2000)}$

$2000 = \ln\left|0+\sqrt{(0)^2+2}\right| + C$

$2000 = \ln\left|0+\sqrt{2}\right| + C$

$2000 = \ln\left|\sqrt{2}\right| + C$

$2000 - \ln\sqrt{2} = C$

$1999.65 \approx C$

$C(x) = \ln\left|x+\sqrt{x^2+2}\right| + 1999.65$

#8) Find the total cost from producing 20 bags of Computer Chips.

$C(x) = \ln\left|x+\sqrt{x^2+2}\right| + 1999.65$

$C(20) = \ln\left|20+\sqrt{(20)^2+2}\right| + 1999.65$

$= \ln\left|20+\sqrt{400+2}\right| + 1999.65$

$= \ln\left|20+\sqrt{402}\right| + 1999.65$

$C(20) \approx 2003.34$

The total cost of producing the first 20 potato computer chips is about $2003.34