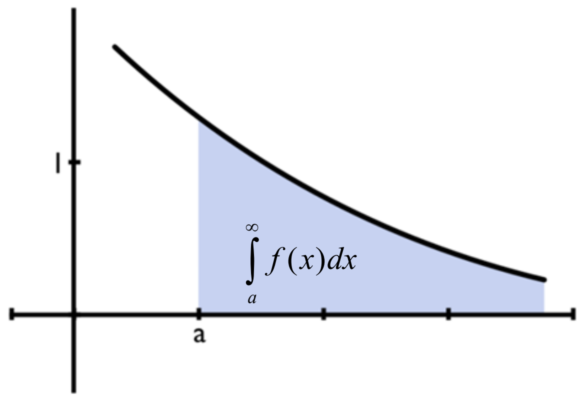
Limits & Improper Integrals

11.3 – Improper Integrals: Integrating to ∞

Improper Integrals – Integrating to ∞

For any continuous and nonnegative function *f* when $x \geq a$, we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$



provided that the limit exists.

Convergent or Divergent?

If the limit exists, it is *convergent.*
If the limit does not exist, it is *divergent*.

Improper Integral

Improper integral = Integrating with infinity or negative infinite.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

<u>Procedure:</u>

#1. Find the area under the curve from a to *b*.

#2. Evaluate the $\lim_{b\to\infty}\int_a^b f(x)\,dx$ to find the area arbitrarily far to the right.

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

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A: Evaluating an Improper Integral

#1) Evaluate $\int_1^\infty \frac{1}{x^2}dx$ and determine if it is convergent or divergent.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$$

$$= \lim_{b\to\infty}\left[-x^{-1}\right]\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\frac{-1}{x}\right]\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\left(-\frac{1}{b}\right)-\left(-\frac{1}{1}\right)\right]$$

$$= \lim_{b\to\infty}\left[-\frac{1}{b}+1\right]$$

$$= -\frac{1}{\infty}+1$$

$$= 0+1$$

$$= 1 \quad (\text{Convergent})$$

To integrate to infinity, first integrate over a *finite* interval, from 1 to some number b.

Then take the limit as *b* approaches . ∞

#2) Evaluate $\int_2^\infty \frac{1}{x^3}dx$

$$\int_2^\infty \frac{1}{x^3}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^3}dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\right]\Big|_2^b$$

$$= \lim_{b\to\infty}\left[\frac{-1}{2x^2}\right]\Big|_2^b$$

$$= \lim_{b\to\infty}\left[\left(\frac{-1}{2(b)^2}\right)-\left(\frac{-1}{2(2)^2}\right)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1}{2b^2}+\frac{1}{8}\right]$$

$$= \frac{-1}{2(\infty)^2}+\frac{1}{8}$$

$$= 0+\frac{1}{8}$$

$$= \frac{1}{8} \quad (\text{Convergent})$$

#3) Evaluate $\int_{16}^{\infty} \frac{1}{\sqrt{x}}\,dx$

$$\int_{16}^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_{16}^{b} x^{-1/2}\,dx$$

$$= \lim_{b\to\infty} \left[2x^{\frac{1}{2}}\right]\Big|_{16}^{b}$$

$$= \lim_{b\to\infty} \left[2\sqrt{x}\right]\Big|_{16}^{b}$$

$$= \lim_{b\to\infty} \left[(2\sqrt{b}) - (2\sqrt{16})\right]$$

$$= 2\sqrt{\infty} - 2\cdot 4$$

$$= \infty - 8$$

$= \infty$, dne divergent

Permanent Endowments

Funds that generate steady income forever are called *permanent endowments.*

George's Tribute – To Himself

#1) George decides to selflessly donate a statue of himself to his beloved home village of Gnadenhutten. Reluctantly, the mayor of Gnadenhutten accepted the statue of George on one condition: George must pay for the upkeep of the statue for eternity. After contacting his accountant, Christopher Reeves, the mayor tells George it will cost $1000 to upkeep the statue per year. Assuming George can gain interest at 5% compounded continuously, the size of the fund necessary to generate $1000 annually forever is $\int_0^{\infty} 1000e^{-0.05t}dt$. Find the size of this permanent endowment by evaluating the integral.

$$\text{Endowment} = \int_0^{\infty} 1000e^{-0.05t}\,dt$$

$$= \lim_{b\to\infty} \int_0^{b} 1000e^{-0.05t}\,dt$$

$$= \lim_{b\to\infty} \left[1000(-20)\,e^{-0.05t}\right]_0^{b}$$

$$= \lim_{b\to\infty} \left[\frac{-20{,}000}{e^{0.05t}}\right]\Big|_0^{b}$$

$$= \lim_{b\to\infty} \left[\left(\frac{-20{,}000}{e^{0.05(b)}}\right) - \left(\frac{-20{,}000}{e^{0.05(0)}}\right)\right]$$

$$= \frac{-20{,}000}{e^{0.05(\infty)}} + \frac{20{,}000}{e^{0}}$$

$$= \frac{20{,}000}{e^{\infty}} + 20{,}000$$

$$= 0 + 20{,}000$$

Endowment = $20,000 (Divergent)

The permanent endowment that will pay $1000 forever is $20,000

Ex. B: Finding an Improper Integral Using a Substitution

#1) Evaluate $\int_2^\infty \frac{2x}{(x^2+1)^2}\,dx$

$$\int_2^\infty \frac{2x}{(x^2+1)^2}\,dx$$

$$u = x^2 + 1$$

$$du = 2x\,dx$$

$$\frac{du}{2x} = dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{2x}{(x^2+1)^2}\,dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{2x}{u^2}\left(\frac{du}{2x}\right)$$

$$= \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{du}{u^2}$$

$$= \lim_{b\to\infty} \left[-u^{-1}\right]\Big|_{x=2}^{x=b}$$

$$= \lim_{b\to\infty} \left[\frac{-1}{x^2+1}\right]\Big|_2^b$$

$$= \lim_{b\to\infty} \left[\left(\frac{-1}{b^2+1}\right) - \left(\frac{-1}{(2)^2+1}\right)\right]$$

$$= \frac{-1}{\infty^2+1} + \frac{1}{4+1}$$

$$= \frac{-1}{\infty} + \frac{1}{5}$$

$$= 0 + \frac{1}{5}$$

$$= \frac{1}{5} \quad (\text{Convergant})$$