Limits & Improper Integrals

11.5 – Improper Integrals: Integrating from $-\infty$ to ∞

Improper Integrals – Integrating to $-\infty$ to ∞

For f continuous and nonnegative for all values of x, we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)dx + \lim_{b\to\infty}\int_{0}^{b} f(x)dx$$

The improper integral is *convergent* if both limits exist, and *divergent* if either limit does not exist.

Ex A: Integrating from $-\infty$ to ∞

Evaluate $\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^3}dx$

$u = 1+e^{-x}$
$du = -e^{-x}dx$
$\frac{du}{-e^{-x}} = dx$

$$= \lim_{a\to-\infty}\int_{a}^{0}\frac{e^{-x}}{(1+e^{-x})^3}dx + \lim_{b\to\infty}\int_{0}^{b}\frac{e^{-x}}{(1+e^{-x})^3}dx$$

$$= \lim_{a\to-\infty}\int_{u=a}^{u=0}\frac{e^{-x}}{u^3}\left(\frac{du}{-e^{-x}}\right) + \lim_{b\to\infty}\int_{u=0}^{u=b}\frac{e^{-x}}{u^3}\left(\frac{du}{-e^{-x}}\right)$$

$$= \lim_{a\to-\infty}\int_{u=a}^{u=0}-u^{-3}du + \lim_{b\to\infty}\int_{u=0}^{u=b}-u^{-3}du$$

$$= \lim_{a\to-\infty}\frac{1}{2}u^{-2}\Big|_{u=a}^{u=0} + \lim_{b\to\infty}\frac{1}{2}u^{-2}\Big|_{u=0}^{u=b}$$

$$= \lim_{a\to-\infty}\frac{1}{2(1+e^{-x})^2}\Big|_{a}^{0} + \lim_{b\to\infty}\frac{1}{2(1+e^{-x})^2}\Big|_{0}^{b}$$

$$= \lim_{a\to-\infty}\left[\frac{1}{2(1+e^{0})^2} - \frac{1}{2(1+e^{-a})^2}\right] + \lim_{b\to\infty}\left[\frac{1}{2(1+e^{-b})^2} - \frac{1}{2(1+e^{0})^2}\right]$$

$$= \frac{1}{2(1+e^{0})^2} - \frac{1}{2(1+e^{-(-\infty)})^2} + \frac{1}{2(1+e^{-(\infty)})^2} - \frac{1}{2(1+e^{0})^2}$$

$$= \frac{1}{2(1+1)^2} - \frac{1}{2(1+e^{\infty})^2} + \frac{1}{2(1+\frac{1}{e^{\infty}})^2} - \frac{1}{2(1+1)^2}$$

$$= \frac{1}{2(2)^2} - \frac{1}{\infty} + \frac{1}{2(1+0)^2} - \frac{1}{2(2)^2}$$

$$= \frac{1}{8} - 0 + \frac{1}{2(1)^2} - \frac{1}{8}$$

$$= \frac{1}{2}$$