Advanced Derivative Rules
4.1 – The Chain Rule

The Chain Rule
If h and k are functions of x, then
$$[h(k(x))]' = h'(k(x)) \cdot k'(x)$$

Prologue

In order to use the Chain Rule, we must first have a composite function. That is, we must have a function inside a function. Let's review how to compose two functions and then how to decompose two functions.

Compose the functions by finding $h(k(x))$.

#1) $h(x) = x^3$
$k(x) = x + 1$

$h(k(x)) = (k(x))^3$
$= (x+1)^3$

#2) $h(x) = x^2$
$k(x) = \frac{1}{x}$

$h(k(x)) = (k(x))^2$
$= \left(\frac{1}{x}\right)^2$
$= \frac{1^2}{x^2}$
$= \frac{1}{x^2}$

Decompose the functions by finding functions $h(x)$ and $k(x)$ such that the following expression is the composition $h(k(x))$.

#1) $(x^3 + 9)^{10}$

inside $= k(x) = x^3 + 9$

outside $= h(x) = x^{10}$

#2) $\sqrt{x^3 + x^2 - 1}$

inside $= k(x) = x^3 + x^2 - 1$

outside $= h(x) = \sqrt{x}$

Ex A: Find each derivative

#1) $f(x) = (x^2 - 4x + 1)^5$

$f'(x) = 5(x^2 - 4x + 1)^4 \cdot (x^2 - 4x + 1)'$
$= 5(x^2 - 4x + 1)^4 (2x - 4)$

#2) $y = (x^4 + x^2 + 8)^6$

$y' = 6(x^4 + x^2 + 8)^5 (x^4 + x^2 + 8)'$
$y' = 6(x^4 + x^2 + 8)^5 (4x^3 + 2x)$

#3) $g(x) = (5x^2 + x)^{10}$

$g'(x) = 10(5x^2 + x)^9 (5x^2 + x)'$
$g'(x) = 10(5x^2 + x)^9 (10x + 1)$

#4) $y = \left(\frac{1}{x^2 - 1}\right)^3 = (x^2 - 1)^{-3}$

$y' = -3(x^2 - 1)^{-4} (x^2 - 1)'$
$y' = -3(x^2 - 1)^{-4} (2x)$
$y' = \frac{-6x}{(x^2 - 1)^4}$

#5) $h(x) = \sqrt{x^2 - 10x + 5}$

$h'(x) = \frac{1}{2}(x^2 - 10x + 5)^{-\frac{1}{2}} (x^2 - 10x + 5)'$
$= \frac{1}{\sqrt{x^2 - 10x + 5}} (2x - 10)$
$= \frac{2x - 10}{\sqrt{x^2 - 10x + 5}}$

Giant Ball of Oil
A giant ball of oil was dropped from the 10^{th} floor of Kramerica Industries. Upon impacting the ground, the oil began to expand on the ground in a circular fashion. After t days of impacting the ground, the radius of the oil slick is $r(t) = \sqrt{8t+1}$ feet. How fast is the radius of the oil slick expanding after 1 day?

r = miles
t = days

$r(t) = (8t+1)^{1/2}$ FIND $\left.\frac{dr}{dt}\right|_{t=1}$

$$\frac{dr}{dt} = \frac{1}{2}(8t+1)^{-\frac{1}{2}}(8t+1)'$$

$$= \frac{1}{2\sqrt{8t+1}}(8)$$

$$= \frac{8}{2\sqrt{8t+1}}$$

$$\frac{dr}{dt} = \frac{4}{\sqrt{8t+1}}$$

$$\left.\frac{dr}{dt}\right|_{t=1} = \frac{4}{\sqrt{8(1)+1}}$$

$$= \frac{4}{\sqrt{8+1}}$$

$$= \frac{4}{\sqrt{9}}$$

$$\left.\frac{dr}{dt}\right|_{t=1} = \frac{4}{3} \text{ miles/day}$$

One day after the ball of oil was dropped, the radius of the oil slick is expanding by 4/3 of a mile per day.