Graphing & Basic Optimization
5.1 – Graphing Using Derivatives

We learned previously that the derivative of a function gives the ___Slope___ of the graph.

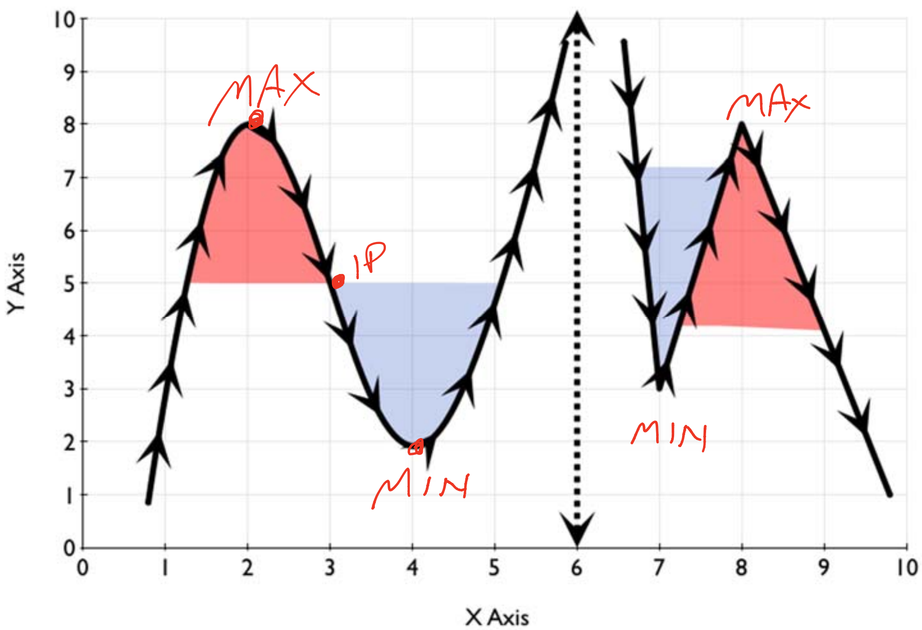
Critical Value: an x-value that changes the nature of a curve. A critical value of function g is an x-value in the domain of g that satisfies one of the following:

$g'(x) = 0$ $\quad\quad$ $g''(x) = 0$
$g'(x)$ is undefined $\quad\quad$ $g''(x)$ is undefined

A critical value can produce a minimum point, a maximum point, an inflection point, or a vertical asymptote.

Critical Point: a point on a graph that changes the nature of the graph.

Relative Maximum Point: a point that is at least as *high* as the points relative to it on the curve on either side.
Relative Minimum Point: a point that is at least as *low* as the points relative to it on the curve on either side.
Inflection Point: a point on a graph that changes the concavity.



A minimum changes the graph from decreasing to increasing.

A maximum change the graph from increasing to decreasing.

An inflection point changes the concavity of the graph.

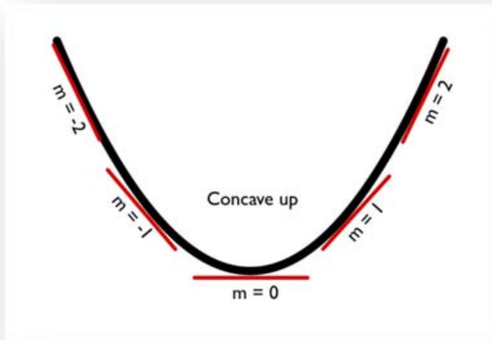
A vertical asymptote can change concavity and can change whether its increasing or decreasing.

Graphing via Derivatives

Step 1: Find CVs by finding when $f' = 0$ (ZON) and $f' = und$ (ZOD)
Step 2: Find CPs by evaluating $f(CV)$
Step 3: Make sign diagram for f'
Step 4: Find CVs by finding when $f'' = 0$ (ZON) and $f'' = und$(ZOD)
Step 5: Find CPs by evaluating $f(CV)$
Step 6: Make sign diagram for f''
Step 7: Find the y-intercept

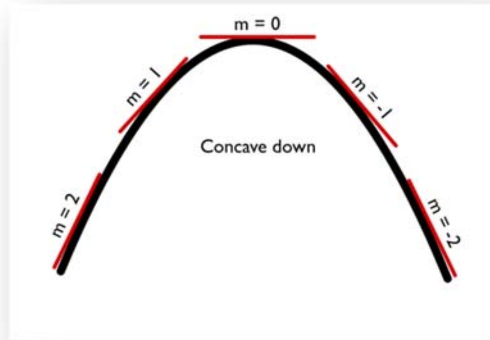
Graphing & Basic Optimization

5.1 – Graphing Using Derivatives



Interpretation of graph:

The slope is increasing, thus $f'' > 0$ means concave up.



Interpretation of graph:

The slope is decreasing, thus $f'' < 0$ means concave down.

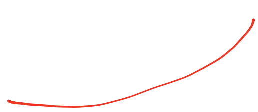
Slope and Concavity

Distinguish carefully between slope and concavity: Slope measures steepness; concavity measures curl.

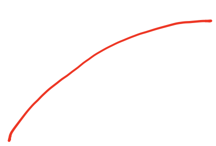
$f' > 0$ is increasing	$f'' > 0$ is concave up
$f' < 0$ is decreasing	$f'' < 0$ is concave down

Ex A: Draw part of a curve described below.

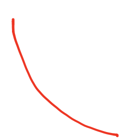
#1) Increasing and concave up



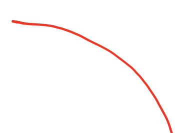
#2) Increasing and concave down



#3) Decreasing and concave up



#4) Decreasing and concave down



Graphing & Basic Optimization

5.1 – Graphing Using Derivatives

Ex B: Guided Example (For day 1, do steps 1 – 3 and 7. For day 2, do steps 4 – 6.)

#1) A company's annual profit after x years is $f(x) = x^3 - 12x^2 - 60x + 15$ million dollars (for $x \geq 0$). Graph this function, show all relative extreme points and inflection points. Interpret the inflection point.

Step 1: Find CVs by finding when $f' = 0$ (ZON) and $f' = und$ (ZOD)

CV

$f'(x) = 3x^2 - 24x - 60$

$0 = 3(x^2 - 8x - 20)$

$0 = 3(x - 10)(x + 2)$

ZON

$0 \neq 3$; $0 = x - 10$, $10 = x$; $0 = x + 2$, $-2 = x$

CV: $x = -2, 10$

Step 2: Find CPs by evaluating $f(CV)$

CP

$f(-2) = 79$

$f(10) = -785$

CP: $(-2, 79)$, $(10, -785)$

Step 3: Make sign diagram for f'

$f'(x) = 3(x - 10)(x + 2)$

$f'(x) = (+)(x - 10)(x + 2)$

$(+)(-)(-) = +$ | $(+)(-)(+) = -$ | $(+)(+)(+) = +$

$f' > 0$ | $f' < 0$ | $f' > 0$

-3 $f'(-2) = 0$ 0 $f'(10) = 0$ 11

MAX $(-2, 79)$ MIN $(10, -785)$

Step 7: Find the y-intercept

y-int

$f(0) = 15$

Step 4: Find CVs by finding when $f'' = 0$ (ZON) and $f'' = und$ (ZOD)

CV

$f''(x) = 6x - 24$

$0 = 6x - 24$

$0 = 6(x - 4)$

ZON

$0 \neq 6$; $0 = x - 4$, $4 = x$

CV: $x = 4$

Step 5: Find CPs by evaluating $f(CV)$

CP

$f(4) = -353$

CP: $(4, -353)$

Step 6: Make sign diagram for f''

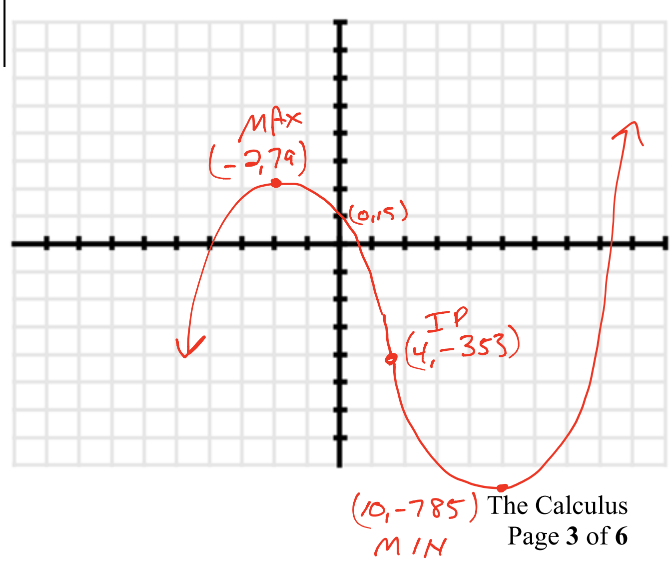
$f''(x) = 6(x - 4)$

$(+)(-) = -$ | $(+)(+) = +$

$f'' < 0$ | $f'' > 0$

3 $f''(4) = 0$ 5

Concave down   IP $(4, -353)$ Concave up



Graphing & Basic Optimization

5.1 – Graphing Using Derivatives

(For day 1, do steps 1 – 3 and 7. For day 2, do steps 4 – 6.)

#2) $f(x) = -x^4 + 4x^3 - 5$

Step 1: CV

$f'(x) = -4x^3 + 12x^2$

$0 = -4x^2(x-3)$

ZON

$0 = -4x^2$, $0 = x^2$, $0 = x$; $0 = x - 3$, $3 = x$

CV: $x = 0, 3$

Step 2: CP

$f(0) = -5$

$f(3) = 22$

CP: $(0,-5), (3,22)$

Step 3:

$0 = -4x^2(x-3)$

$0 = (-)(+)(x-3)$

$(-)(+)(-)=+$ | $(-)(+)(-)=+$ | $(-)(+)(+)=-$

$f'>0$ | $f'>0$ | $f'<0$

-1 $f'(0)=0$ 1 $f'(3)=0$ 4

Neither MAX $(3,22)$

Step 7:

y-int

$f(0) = -5$

Step 4: CV

$f''(x) = -12x^2 + 24x$

$0 = -12x(x-2)$

ZON

$0 = -12x$, $0 = x$; $0 = x - 2$, $2 = x$

CV: $x = 0, 2$

Step 5: CP

$f(0) = -5$

$f(2) = 11$

CP: $(0,-5), (2,11)$

Step 6:

$f'(x) = -12x(x-2)$

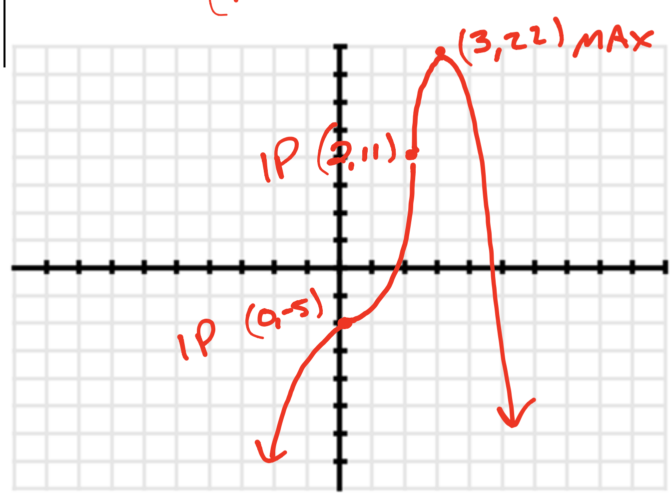
$f'(x) = (-)x(x-2)$

$(-)(-)(-)=-$ | $(-)(+)(-)=+$ | $(-)(+)(+)=-$

$f''<0$ | $f''>0$ | $f''<0$

-1 $f''(0)=0$ 1 $f''(2)=0$ 3

Concave down IP $(0,-5)$ Concave up IP $(2,11)$ Concave down



Graphing & Basic Optimization

5.1 – Graphing Using Derivatives

(For day 1, do steps 1 – 3 and 7. For day 2, do steps 4 – 6.)

#3) $f(x) = 9\sqrt[3]{(x-1)^2} = 9(x-1)^{2/3}$

#1) CV

$f'(x) = 6(x-1)^{-1/3}$

$0 = \dfrac{6}{\sqrt[3]{x-1}}$

ZON

$0 \neq 6$

ZOD

$0 = \sqrt[3]{x-1}$

$0 = x-1$

$1 = x$

CV: $x = 1$

#2) CP

$f(1) = 0$

CP: $(1, 0)$

#3)

$f'(x) = \dfrac{6}{\sqrt[3]{x-1}}$

$f'(x) = \dfrac{(+)}{\sqrt[3]{x-1}}$

$\frac{+}{-} = -$ | $\frac{+}{+} = +$

$f' < 0$ | $f' > 0$

0 $f'(1) = \text{und}$ 2

MIN $(1,0)$

#7 y-int

$f(0) = 9$

#4) CV

$f''(x) = -2(x-1)^{-4/3}$

$0 = \dfrac{-2}{\sqrt[3]{(x-1)^4}}$

Z.O.N.

$0 \neq -2$

Z.O.D.

$0 = \sqrt[3]{(x-1)^4}$

$0 = x - 1$

$x = 1$

CV: $x = 1$

#5) CP

$f(1) = 0$

CP: $(1, 0)$

#6)

$f''(x) = \dfrac{-2}{\sqrt[3]{(x-1)^4}}$

$f''(x) = \dfrac{(-)}{(+)}$

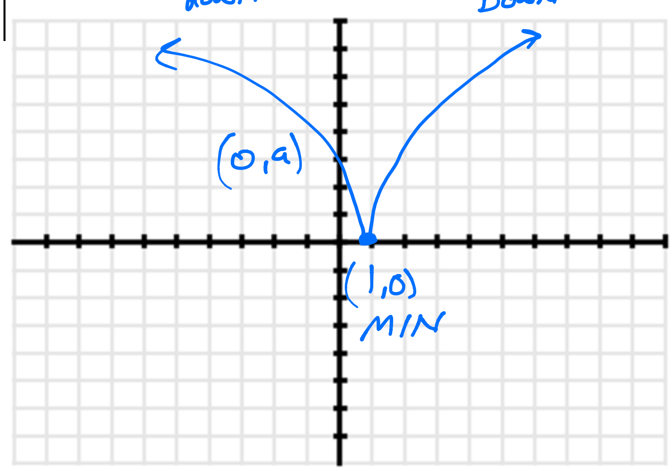
$f''(x) = (-)$

$(-)$ | $(-)$

$f'' < 0$ | $f'' < 0$

0 $f''(1) = \text{und}$ 2

Concave down | Concave Down



(For day 1, do steps 1 – 3 and 7. For day 2, do steps 4 – 6.)

#4) $f(x) = \frac{2}{x^2-4x} = \frac{2}{x(x-4)} = 2(x^2-4x)^{-1}$

RATIONAL FUNCTION!

If you have to graph a rational function, FIND holes and Asymptotes. You DONT have to do the second derivative sign diagram.

Holes & Asymptotes

VA

$x(x-4)=0$

$x=0$, $x-4=0$, $x=4$

HA

$n?d$

$0<2$, so HA $y=0$

#1) CV

$f(x) = 2(x^2-4x)^{-1}$

$f'(x) = -2(x^2-4x)^{-2}(x^2-4x)'$

$f'(x) = -2(x^2-4x)^{-2}(2x-4)$

$0 = \frac{-4(x-2)}{(x^2-4x)^2}$

ZON

$0 = -4(x-2)$

$0 \neq -4$, $0 = x-2$, $2 = x$

CV: $x=2$

ZOD

$0 = (x^2-4x)^2$

$0 = x^2-4x$

$0 = x(x-4)$

$0 = x$, $0 = x-4$, $4 = x$

CV: $x = 0, 4$

#2) CP

$f(0) = \text{und}$

$f(2) = -\frac{1}{2}$

$f(4) = \text{und}$

CP: $(2, -\frac{1}{2})$

#3)

$f'(x) = \frac{-4(x-2)}{(x^2-4x)^2}$

$f'(x) = \frac{(-)(x-2)}{+}$

$f'(x) = (-)(x-2)$

$(-)(-)=+$ | $(-)(-)=+$ | $(-)(+)=-$ | $(-)(+)=-$

$f'>0$ | $f'>0$ | $f'<0$ | $f'<0$

-1 | $f'(0)=\text{und}$ | 1 | $f'(2)=0$ | 3 | $f'(4)=\text{und}$ | 5

VA | MAX $(2,-\frac{1}{2})$ | VA

#7) Y-int

$\boxed{f(0) = \text{und}}$