# Advanced Techniques
## 6.4A – Related Rates

**Hint: the volume of a sphere exists ☺**

**Chocolate**

#1) George left his large ball of chocolate in the sun so that its radius is decreasing at the rate of 2 inches per minute. How fast is the volume decreasing at the moment when the radius is 3 inches?

$\frac{dr}{dt} = \frac{-2\text{ in}}{1\text{ min}}$  FIND $\left.\frac{dV}{dt}\right|_{r=3}$

r = radius (inches)
t = time (minutes)
V = Volume (in³)

$$V_{\circledast} = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2(-2)$$

$$\frac{dV}{dt} = -8\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{r=3} = -8\pi(3)^2$$

$$= -8\pi(9)$$

$$= -72\pi$$

$$\left.\frac{dV}{dt}\right|_{r=3} \approx -226.2 \text{ in}^3/\text{min}$$

*Sentence Answer:*

when the radius of the ball of choc is 3 inches, the volume is decreasing by 226.2 in³ per minute.

**Gallstones**

#2) A gallstone is forming in George's gallbladder so that its radius is growing at the rate of 1 centimeter per year. How fast is its volume growing at the moment when the radius is 2 centimeters?

$\frac{dr}{dt} = \frac{1\text{ cm}}{1\text{ yr}}$  FIND $\left.\frac{dV}{dt}\right|_{r=2}$

r = radius (cm)
t = time (year)
V = Volume (cm³)

$$V_{\circledast} = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2(1)$$

$$\frac{dV}{dt} = 4\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{r=2} = 4\pi(2)^2$$

$$= 4\pi \cdot 4$$

$$= 16\pi$$

$$\left.\frac{dV}{dt}\right|_{r=2} \approx 50.3 \text{ cm}^3/\text{year}$$

*Sentence Answer:*

when the radius of the gallstone is 2 cm, the volume is increasing by 50.3 cm³ per year.

**Gumball**

#3) George is entering a chewed gumball contest. The radius of his spherical gumball is growing by ½ centimeter per week. Find how rapidly the volume is increasing at the moment when the radius is 4 centimeters.

$\frac{dr}{dt} = \frac{\frac{1}{2}\text{ cm}}{1\text{ week}}$

FIND $\left.\frac{dV}{dt}\right|_{r=4}$

$r$ = radius (cm)
$t$ = time (week)
$V$ = volume (cm$^3$)

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dr} = 4\pi r^2\left(\frac{dr}{dt}\right)$$

$$\frac{dV}{dr} = 4\pi r^2\left(\frac{1}{2}\right)$$

$$\frac{dV}{dr} = 2\pi r^2$$

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$$\left.\frac{dV}{dr}\right|_{r=4} = 2\pi(4)^2$$

$$= 2\pi(16)$$

$$= 32\pi$$

$$\left.\frac{dV}{dr}\right|_{r=4} \approx 100.5 \text{ cm}^3/\text{week}$$

*Sentence Answer:*

When the radius of the gum is 4 inches, the volume is increasing by 100.5 in$^3$ per week

**Pig Feet**

#4) George's profit from selling $x$ boxes pig feet is $P = 1000x - \frac{1}{2}x^2$ dollars. If sales are growing at the rate of 20 per day, find how rapidly profit is growing (in dollars per day) when 600 boxes have been sold.

$x$ = Sales (boxes of pig feet)
$P$ = Profit \$
$t$ = time (day)

$\frac{dx}{dt} = \frac{20\text{ boxes}}{1\text{ day}}$

FIND $\left.\frac{dP}{dt}\right|_{x=600}$

$$P = 1000x - \frac{1}{2}x^2$$

$$\frac{d}{dt}P = \frac{d}{dt}(1000x) - \frac{d}{dt}\left(\frac{1}{2}x^2\right)$$

$$\frac{dP}{dt} = 1000\frac{dx}{dt} - x\frac{dx}{dt}$$

$$\frac{dP}{dt} = 1000(20) - x(20)$$

$$\frac{dP}{dt} = 20{,}000 - 20x$$

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$$\left.\frac{dP}{dt}\right|_{x=600} = 20{,}000 - 20(600)$$

$$= 20{,}000 - 12{,}000$$

$$\left.\frac{dP}{dt}\right|_{x=600} = 8{,}000$$

*Sentence Answer:*

When 600 boxes of pigs feet have been sold, the profit is increasing by \$8000 per feet

**Tons of Feathers**

#5) George's revenue from selling x tons of feathers is given as $R = 1000x - x^2$ dollars. If sales are increasing at the rate of 80 per day, find how rapidly revenue is growing (in dollars per day) when 400 tons have been sold.

x = sales (tons of feathers)
R = Revenue \$
t = time (days)

$\frac{dx}{dt} = \frac{80 \text{ tons}}{1 \text{ day}}$

FIND $\left.\frac{dR}{dt}\right|_{x=400}$

$$R = 1000x - x^2$$

$$\frac{d}{dt}R = \frac{d}{dt}(1000x) - \frac{d}{dt}(x^2)$$

$$\frac{dR}{dt} = 1000\frac{dx}{dt} - 2x\frac{dx}{dt}$$

$$\frac{dR}{dt} = 1000(80) - 2x(80)$$

$$\frac{dR}{dt} = 80{,}000 - 160x$$

$$\left.\frac{dR}{dt}\right|_{x=400} = 80{,}000 - 160(400)$$

$$= 80{,}000 - 64{,}000$$

$$\left.\frac{dR}{dt}\right|_{x=400} = \$16{,}000/\text{day}$$

*Sentence Answer:*

When 400 tons of feathers have been sold, the revenue is increasing by \$16,000 per day.

**Accidents**

#6) The number of traffic accidents George's stench causes per year in a population p is predicted to be $T = 0.002p^{3/2}$. If the population is growing by 500 people per year, find the rate at which traffic accidents will be rising when the population is 40,000.

p = population (people)
T = traffic accidents
t = time (year)

$\frac{dP}{dt} = \frac{500 \text{ people}}{1 \text{ year}}$

FIND $\left.\frac{dT}{dt}\right|_{p=40{,}000}$

$$T = 0.002p^{3/2}$$

$$\frac{d}{dt}T = \frac{d}{dt}\left(0.002p^{3/2}\right)$$

$$\frac{dT}{dt} = 0.003p^{\frac{1}{2}}\frac{dp}{dt}$$

$$\frac{dT}{dt} = 0.003\sqrt{p}\,(500)$$

$$\frac{dT}{dt} = 1.5\sqrt{p}$$

$$\left.\frac{dT}{dt}\right|_{p=40{,}000} = 1.5\sqrt{40{,}000}$$

$$= 1.5(200)$$

$$\left.\frac{dT}{dt}\right|_{p=40{,}000} = 300 \text{ Traffic accidents/year}$$

*Sentence Answer:*

When the pop'n is 40,000 people, the number of accidents George's stench is causing is increasing by 300 accidents per year

**Carnival Slaying**

#7) George is a carny and witnesses many types of crimes. The number of slayings at George's carnival of population p is expected to be $W = 0.003p^{4/3}$. If the population is growing by 1000 people per year, find the rate at which the number of carnival slayings will be increasing when the population is 1,000,000.

P = population of carnival
W = # of slayings
t = time in years

$\frac{dP}{dt} = \frac{1000 \text{ people}}{1 \text{ year}}$

$\left.\frac{dW}{dt}\right|_{P=1{,}000{,}000}$

$$W = 0.003P^{4/3}$$

$$\frac{d}{dt}(W) = \frac{d}{dt}\left(0.003P^{4/3}\right)$$

$$\frac{dW}{dt} = 0.004P^{1/3}\frac{dP}{dt}$$

$$\frac{dW}{dt} = 0.004\sqrt[3]{P}\,(1000)$$

$$\frac{dW}{dt} = 4\sqrt[3]{P}$$

$$\left.\frac{dW}{dt}\right|_{P=1{,}000{,}000} = 4\sqrt[3]{1{,}000{,}000}$$

$$= 4(1000)$$

$$\left.\frac{dW}{dt}\right|_{P=1{,}000{,}000} = 4000 \text{ slayings per year}$$

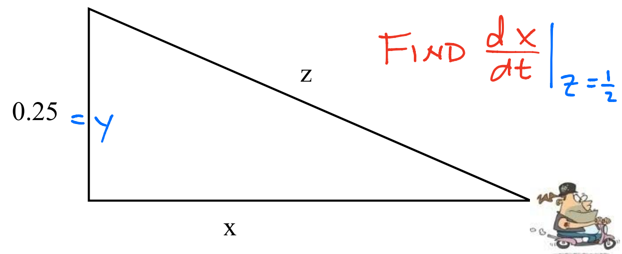
*Sentence Answer:*

When the carnival pop'n is 1,000,000, the number of carnival slayings is increasing by 4000 slayings per year.

**Speeding**

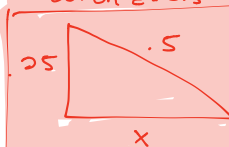
#8) A traffic patrol helicopter is stationary a quarter of a mile directly above a highway, as shown in the diagram below. Its radar detects George's moped whose line-of-sight distance from the helicopter is half a mile and is increasing at the rate of 57 mph. Is the moped exceeding the highway's speed limit of 60 mph?

$\frac{dz}{dt} = \frac{57 \text{ mi}}{1 \text{ hr}}$




FIND $\left.\frac{dx}{dt}\right|_{z=\frac{1}{2}}$

when z = 0.5




$(.25)^2 + x^2 = (0.5)^2$

$.0625 + x^2 = .25$

$x^2 = .1875$

$x \approx .433$

$$x^2 + y^2 = z^2$$

$$x^2 + (0.25)^2 = z^2$$

$$x^2 + 0.625 = z^2$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(0.625) = \frac{d}{dt}(z^2)$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{z}{x}(57)$$

$$\left.\frac{dx}{dt}\right|_{z=\frac{1}{2},\ x=.433} \approx \frac{\left(\frac{1}{2}\right)}{(.433)}\,57$$

$$\approx 65.8 \text{ mph}$$

*Sentence Answer:*

When George is ½ mile from the chopper, he is breaking the speed limit by traveling at 65.8 miles per hour.

#1) At the moment the radius is 3 inches, the volume is decreasing by $72\pi \approx 226\ in^3$ in per hour.

#2) When the radius is 2 mm, the volume is growing at $16\pi \approx 50.27\ mm^3$ per minute.

#3) When the radius is 4 cm, the volume of the tumor is growing at $32\pi \approx 101\ cm^3$ per weak.

#4) When 600 units have been sold, the profit is growing by $8000 per day.

#5) When 400 units have been sold, the revenue is growing by $16,000 per day.

#6) When the population is 40,000 people, traffic accidents will be rising by 300 accidents per year.

#7) When the population is 1,000,000 people, the number of carnival slayings is increasing by 400 cases per year.

#8) The car is traveling at 65.8 mph, so yes the car is speeding.

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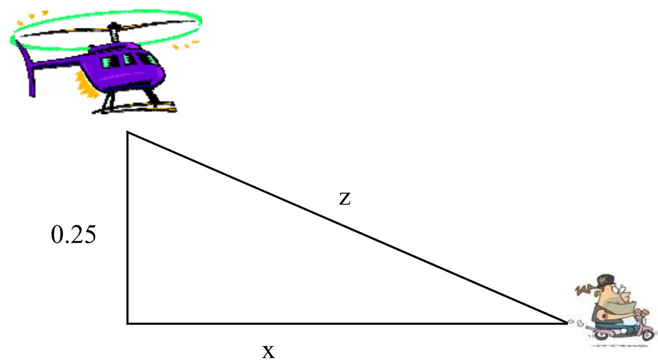
## 6.4A – Related Rates

#8)




<u>Facts</u>:
z = line of sight distance in miles
x = horizontal distance in miles
$\frac{dz}{dt}$ = 57 miles/hour  Find $\frac{dx}{dt}$ evaluated at z = .5 miles

From Pythagoras

$x^2 + 0.25^2 = z^2$
$x^2 + 0.25^2 = 0.5^2$
$x^2 + 0.0625 = 0.25$
$x^2 = .1875$
$x \approx 0.433$

From Pythagoras

$x^2 + 0.25^2 = z^2$

$\frac{d}{dt}x^2 + \frac{d}{dt}0.25^2 = \frac{d}{dt}z^2$  Found $\frac{d}{dt}$

$2x\frac{dz}{dt} + 0 = 2z\frac{dz}{dt}$  Simplified

$2x\frac{dz}{dt} = 2z\frac{dz}{dt}$  Simplified

$\frac{dz}{dt} = \frac{2z}{2x}\frac{dz}{dt}$  Solving for $\frac{dz}{dt}$, I divided by 2x

$= \frac{z}{x}\frac{dz}{dt}$  The 2's canceled, because 2 / 2 = 1

$\frac{dz}{dt}$ evaluated $\approx \frac{0.5}{.433}57$  Substituted $\frac{dz}{dt} = 57$, z = .5, x ≈ 0.433

$\approx \frac{0.5(57)}{.433}$  Simplified

≈ 65.8 miles per hour  Simplified

Yes, the moped is exceeding the highway's speed of 60 mph. George is traveling at a rate of approximately 65.8 mph.