Advanced Techniques
6.4 – Related Rates

Sometimes *both* variables in an equation will be functions of a *third* variable, usually t for time. For example, for a seasonal product such as fur coats, the price p and weekly sales x will be related by a demand equation, and both price p and quantity x will depend on the time of year.

Differentiating both sides of the demand equation with respect to time t will give an equation relating the derivatives $\frac{dp}{dt}$ and $\frac{dx}{dt}$

Such "related rates" equations show how fast one quantity is changing relative to another.

$\frac{dr}{dt} = \frac{2 \text{ feet}}{1 \text{ sec}}$

Ex A: Finding Related Rates

#1) A pebble thrown into a pond causes circular ripples to radiate outward. If the radius of the outer ripple is growing by 2 feet per second, how fast is the area of the circle growing at the moment when the radius is 10 feet?

Pro Tips

FIND $\left.\frac{dA}{dt}\right|_{r=10}$

A = Area in feet²
r = radius in feet
t = time in seconds

Since the radius is increasing by 2 feet per second, $\frac{dr}{dt} = \frac{2 \text{ ft}}{1 \text{ sec}}$

$A = \pi r^2$

$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$\frac{dA}{dt} = 2r\pi \frac{dr}{dt}$

$\frac{dA}{dt} = 2r\pi(2)$

$\frac{dA}{dt} = 4r\pi$

$\left.\frac{dA}{dt}\right|_{r=10} = 4(10)\pi$

$= 40\pi \text{ ft}^2/\text{sec}$

$\left.\frac{dA}{dt}\right|_{r=10} \approx 125.6 \text{ ft}^2/\text{sec}$

When the radius is 10 ft, the area of the circle is growing by 125.6 ft/sec

#2) A boat yard's total profit from selling x outboard motors is $P(x) = -x^2 + 1000x - 2000$. If the outboards are selling at the rate of 20 per week, how fast is the profit changing when 400 motors have been sold?

Pro Tips

FIND $\left.\frac{dP}{dt}\right|_{x=400}$

x = outboard motors
P = profit
t = time in weeks

$\frac{dx}{dt} = \frac{20 \text{ outboards}}{1 \text{ week}}$

$P(x) = -x^2 + 1000x - 2000$

$\frac{d}{dt} P(x) = \frac{d}{dt}(-x^2) + \frac{d}{dt}(1000x) - \frac{d}{dt}(2000)$

$\frac{dP}{dt} = -2x\frac{dx}{dt} + 1000\frac{dx}{dt} - 0$

$\frac{dP}{dt} = -2x\frac{dx}{dt} + 1000\frac{dx}{dt}$

$\frac{dP}{dt} = -2x(20) + 1000(20)$

$\frac{dP}{dt} = -40x + 20000$

$\left.\frac{dP}{dt}\right|_{x=400} = -40x + 20{,}000$

$= -40(400) + 20{,}000$

$= -16{,}000 + 20{,}000$

$\left.\frac{dP}{dt}\right|_{x=400} = \$4{,}000/\text{week}$

When 400 motors have been sold, the profit is growing by $4000 per week.

#3) A study of urban pollution predicts that sulfur oxide emissions in a city will be $S(x) = 0.1x^2 + 20x + 2$ tons, where x is the population (in thousands). The population of the city t years from now is expected to be $x = 20\sqrt{t} + 800$ thousand people. Find how rapidly sulfur oxide pollution will be increasing 4 years from now.

Pro Tips

S = Sulfur oxide

X = pop in thousands

t = years

Find $\left.\frac{dS}{dt}\right|_{t=4}$

$S(x) = 0.1x^2 + 20x + 2$

$\frac{d}{dt} S(x) = \frac{d}{dt}(0.1x^2) + \frac{d}{dt}(20x) + \frac{d}{dt}(2)$

$\frac{dS(x)}{dt} = 0.2x\frac{dx}{dt} + 20\frac{dx}{dt} + 0$

$\frac{dS(x)}{dt} = 0.2x\frac{dx}{dt} + 20\frac{dx}{dt}$

$x = 20\sqrt{t} + 800$

$\frac{dx}{dt} = \frac{d}{dt}(20t^{\frac{1}{2}}) + \frac{d}{dt}(800)$

$\frac{dx}{dt} = 10t^{-\frac{1}{2}}$

$\frac{dx}{dt} = \frac{10}{\sqrt{t}}$

$\frac{dS(x)}{dt} = 0.2(20\sqrt{t} + 800)\frac{dx}{dt} + 20\frac{dx}{dt}$

$\frac{dS}{dt} = (4\sqrt{t} + 160)\frac{dx}{dt} + 20\frac{dx}{dt}$

$\frac{dS}{dt} = (4\sqrt{t} + 160)\left(\frac{10}{\sqrt{t}}\right) + 20\left(\frac{10}{\sqrt{t}}\right)$

$= 40 + \frac{1600}{\sqrt{t}} + \frac{200}{\sqrt{t}}$

$\left.\frac{dS}{dt}\right|_{t=4} = 40 + \frac{1600}{\sqrt{4}} + \frac{200}{\sqrt{4}}$

$= 40 + \frac{1600}{2} + \frac{200}{2}$

$= 40 + 800 + 100$

$= 940$ tons per year

In 4 years, the s.o.e. will increase by 940 ton/week