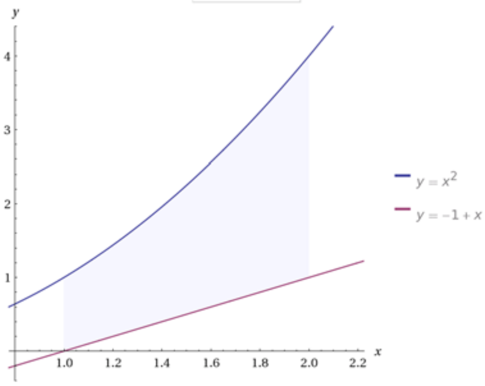
Basic Integration
8.3A – Area Between Curves

A: Find the area between curves that may or may not cross.

#1)

area between	$y = x^2$	domain	$1 \le x \le 2$
	$y = -1 + x$		



① cross? NO ② upper: $y = x^2$ Lower: $y = -1 + x$

③ $A = \int_1^2 [(x^2) - (-1 + x)]\,dx$

$= \int_1^2 [x^2 - x + 1]\,dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]\Big|_1^2$

$= \left[\frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + (2)\right] - \left[\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + (1)\right]$

$= \left[\frac{1}{3}(8) - \frac{1}{2}(4) + 2\right] - \left[\frac{1}{3}(1) - \frac{1}{2}(1) + 1\right]$

$= \left[\frac{8}{3} - 2 + 2\right] - \left[\frac{1}{3} - \frac{1}{2} + 1\right]$

$= \frac{8}{3} - \frac{1}{3} + \frac{1}{2} - 1$

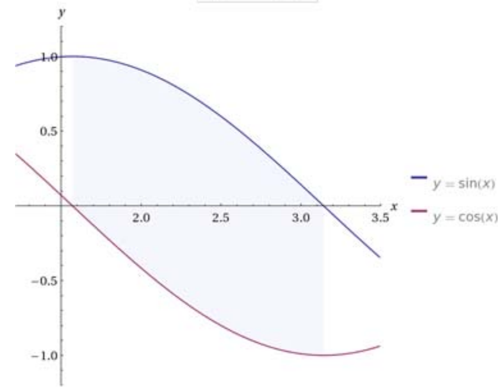
$= \frac{7}{3} - \frac{1}{2}$

$= \frac{14}{6} - \frac{3}{6}$

$A = \frac{11}{6}\ un^2$

#2)

area between	$y = \sin(x)$	domain	$\frac{\pi}{2} \le x \le \pi$
	$y = \cos(x)$		



① cross? NO ② upper: $y = \sin(x)$ Lower: $y = \cos(x)$

③ $A = \int_{\pi/2}^{\pi} [\sin(x) - \cos(x)]\,dx$

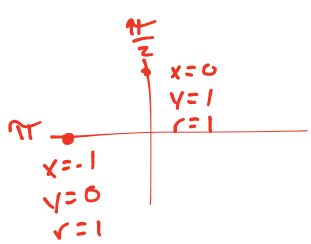
$= \left[-\cos(x) - \sin(x)\right]\Big|_{\pi/2}^{\pi}$

$= \left[-\cos(\pi) - \sin(\pi)\right] - \left[-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right]$

$= [-(-1) - (0)] - [-(0) - (1)]$

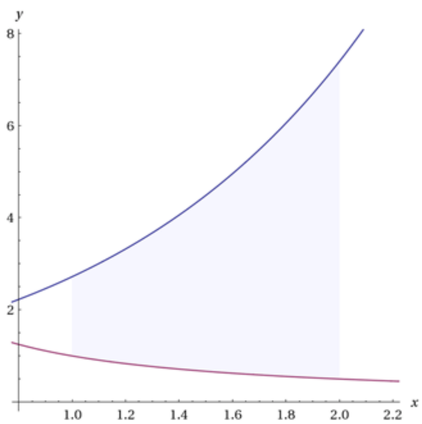
$= [1] - [-1]$

$A = 2\ un^2$



#3)

area between	$y = e^x$, $y = \frac{1}{x}$	domain	$1 \le x \le 2$



① Cross? No ② upper $y = e^x$ Lower $y = \frac{1}{x}$

③ $A = \int_1^2 \left[(e^x) - \left(\frac{1}{x}\right)\right] dx$

$= \left[e^x - \ln|x|\right] \Big|_1^2$

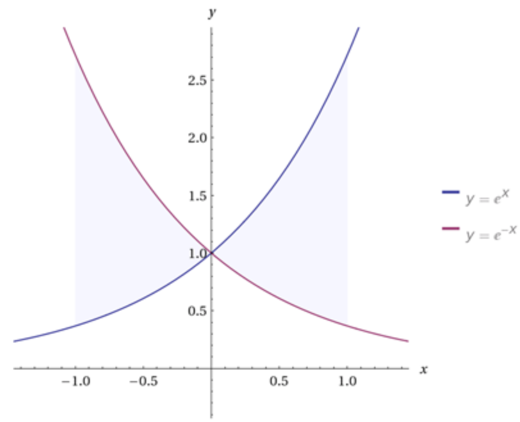
$= \left[e^2 - \ln|2|\right] - \left[e^1 - \ln|1|\right]$

$= e^2 - \ln(2) - e + 0$

$A = \left[e^2 - e - \ln(2)\right] \text{un}^2$

#4)

area between	$y = e^x$, $y = e^{-x}$	domain	$-1 \le x \le 1$



① Cross? Yes @ 0

② $[-1, 0)$ upper: $y = e^{-x}$ Lower: $y = e^x$; $(0, 1]$ $y = e^x$ $y = e^{-x}$

③ $A = \int_{-1}^{0} \left[e^{-x} - e^x\right] dx + \int_0^1 \left[e^x - e^{-x}\right] dx$

$= \left[-e^{-x} - e^x\right] \Big|_{-1}^{0} + \left[e^x + e^{-x}\right] \Big|_0^1$

$= \left[-e^{-(0)} - e^0\right] - \left[-e^{-(-1)} - e^{(-1)}\right] + \left[e^1 + e^{-(1)}\right] - \left[e^0 + e^{-(0)}\right]$

$= [-1 - 1] - \left[-e - \frac{1}{e}\right] + \left[e + \frac{1}{e}\right] - [1 + 1]$

$= [-2] + e + \frac{1}{e} + e + \frac{1}{e} - [2]$

$A = \left(2e + \frac{2}{e} - 4\right) \text{un}^2$

#5)

area between	$y=4+x^2$	domain	$0\le x\le 3$
	$y=1+2x$		

(To determine where they cross, you will need to complete the square.)

① Cross? NO

$4+x^2=1+2x$

$x^2-2x+3=0$

Complete square → $(x^2-2x+1)-1+3=0$

$(x-1)^2+2=0$

$(x-1)^2=-2$

$x-1=\pm\sqrt{-2}$

$x=1\pm\sqrt{-2}$

$x=$ undefined (imaginary)

② Upper / Lower

$0\in[0,3]$

$y=4+x^2$	$y=1+2x$
$y=4+(0)^2$	$y=1+2(0)$
$y=4$	$y=1$
upper	Lower

③ $A=\int_0^3[(4+x^2)-(1+2x)]dx$

$=\int_0^3[x^2-2x+3]dx$

$=\left[\frac{1}{3}x^3-x^2+3x\right]\Big|_0^3$

$=\left[\frac{1}{3}(3)^3-(3)^2+3(3)\right]-\left[\frac{1}{3}(0)^3-(0)^2+3(0)\right]$

$=\left[\frac{1}{3}(27)-9+9\right]-[0]$

$A=9\text{ un}^2$

#6)

area between	$y=3+x^2$	domain	$0\le x\le 3$
	$y=8+2x$		

(To determine where they cross, you will need to complete the square.)

① Cross? Not on $[0,3]$

$3+x^2=8+2x$

$x^2-2x-5=0$

Doesn't FACTOR

Complete square → $(x^2-2x+1)-1-5=0$

$(x-1)^2-6=0$

$(x-1)^2=6$

$x-1=\pm\sqrt{6}$

$x=1\pm\sqrt{6}$

$1\pm\sqrt{6}\notin[0,3]$

② upper / Lower

$0\in[0,3]$

$y=3+x^2$	$y=8+2x$
$y=3+(0)^2$	$y=8+2(0)$
$y=3$	$y=8$
Lower	upper

③ $A=\int_0^3[(8+2x)-(3+x^2)]dx$

$=\int_0^3[-x^2+2x+5]dx$

$=\left[-\frac{1}{3}x^3+x^2+5x\right]\Big|_0^3$

$=\left[-\frac{1}{3}(3)^3+(3)^2+5(3)\right]-\left[-\frac{1}{3}(0)^3+(0)^2+5(0)\right]$

$=\left[-\frac{1}{3}(27)+9+15\right]-[0]$

$=-9+9+15$

$A=15\text{ un}^2$

A: Find the area between curves that may or may not cross.

#7)

area between	$y = 2\sin(x)$	domain	$\frac{\pi}{4} \le x \le \frac{\pi}{2}$
	$y = 1$		

Round to hundredths

① cross? Not in $[\frac{\pi}{4}, \frac{\pi}{2}]$

$1 = 2\sin(x)$

$\frac{1}{2} = \sin(x)$

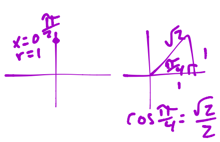
$x = 30°, 150°$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \notin [\frac{\pi}{4}, \frac{\pi}{2}]$

② Upper / Lower

$\frac{\pi}{2} \in [\frac{\pi}{4}, \frac{\pi}{2}]$

$y = 1$	$y = 2\sin(x)$
	$y = 2\sin(\frac{\pi}{2})$
	$y = 2(1)$
	$y = 2$
Lower	upper



③ $A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [2\sin(x) - 1]\,dx$

$= [-2\cos(x) - x]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= [-2\cos(\frac{\pi}{2}) - \frac{\pi}{2}] - [-2\cos(\frac{\pi}{4}) - \frac{\pi}{4}]$

$= [-2(0) - \frac{\pi}{2}] - [-2(\frac{\sqrt{2}}{2}) - \frac{\pi}{4}]$

$= [-\frac{\pi}{2}] - [-\sqrt{2} - \frac{\pi}{4}]$

$= -\frac{2\pi}{4} + \sqrt{2} + \frac{\pi}{4}$

$A = (-\frac{\pi}{4} + \sqrt{2})\ un^2$

$A \approx 0.63\ un^2$

#8)

area between	$y = 1 + 3x^2$	domain	$0 \le x \le 3$
	$y = 2 + 2x$		

① cross

$1 + 3x^2 = 2 + 2x$

$3x^2 - 2x - 1 = 0$

$(3x^2 - 3x) + (x - 1) = 0$

$3x(x-1) + 1(x-1) = 0$

$(x-1)(3x+1) = 0$

$x - 1 = 0$; $3x + 1 = 0$

$x = 1$; $3x = -1$

$1 \in [0,3]$; $x = -\frac{1}{3}$

$-\frac{1}{3} \notin [0,3]$

Upper/Lower $0 \in [0,1)$ — Upper/Lower $2 \in (1,3]$

$y = 1 + 3x^2$	$y = 2 + 2x$	$y = 1 + 3x^2$	$y = 2 + 2x$
$y = 1 + 3(0)^2$	$y = 2 + 2(0)$	$y = 1 + 3(2)^2$	$y = 2 + 2(2)$
$y = 1$	$y = 2$	$y = 1 + 3(4)$	$y = 2 + 4$
		$y = 1 + 12$	$y = 6$
		$y = 13$	
Lower	upper	upper	Lower

③ $A = \int_0^1 [(2+2x) - (1+3x^2)] + \int_1^3 [(1+3x^2) - (2+2x)]\,dx$

$= \int_0^1 [-3x^2 + 2x + 1]\,dx + \int_1^3 [3x^2 - 2x - 1]\,dx$

$= [-x^3 + x^2 + x]\Big|_0^1 + [x^3 - x^2 - x]\Big|_1^3$

$= [-(1)^3 + (1)^2 + (1)] - [-(0)^3 + (0)^2 + (0)] + [(3)^3 - (3)^2 - (3)] - [(1)^3 - (1)^2 - (1)]$

$= [-1 + 1 + 1] - [0] + [27 - 9 - 3] - [1 - 1 - 1]$

$= 1 + [15] - [-1]$

$= 1 + 15 + 1$

$A = 17\ un^2$

B: Find the area bounded by the curves

#9)

area between	$y = x^2$
	$y = 3 - 2x^2$

① cross? yes @ -1 and 1

$x^2 = 3 - 2x^2$
$3x^2 - 3 = 0$
$3(x^2 - 1) = 0$
$3(x-1)(x+1) = 0$
$x - 1 = 0$; $x + 1 = 0$
$x = 1$; $x = -1$
$(-1, 1)$

② upper/Lower
$0 \in (-1, 1)$

$y = x^2$	$y = 3 - 2x^2$
$y = (0)^2$	$y = 3 - 2(0)^2$
$y = 0$	$y = 3 - 0$
	$y = 3$
Lower	upper

③ $A = \int_{-1}^{1} [(3 - 2x^2) - (x^2)]\,dx$

$= \int_{-1}^{1} [-3x^2 + 3]\,dx$

$= [-x^3 + 3x] \Big|_{-1}^{1}$

$= [-(1)^3 + 3(1)] - [-(-1)^3 + 3(-1)]$

$= [-(1) + 3] - [-(-1) - 3]$

$= [2] - [-2]$

$A = 4 \text{ un}^2$

#10)

area between	$y = -10x + 6x^2$
	$y = -15 + 8x + 3x^2$

① cross? Yes @ 1 and 5

$-10x + 6x^2 = -15 + 8x + 3x^2$
$3x^2 - 18x + 15 = 0$
$3[x^2 - 6x + 5] = 0$
$3(x-5)(x-1) = 0$
$x - 5 = 0$; $x - 1 = 0$
$x = 5$; $x = 1$
$(1, 5)$

② upper/Lower
$2 \in (1, 5)$

$y = -10x + 6x^2$	$y = -15 + 8x + 3x^2$
$y = -10(2) + 6(2)^2$	$y = -15 + 8(2) + 3(2)^2$
$y = -20 + 6(4)$	$y = -15 + 16 + 3(4)$
$y = -20 + 24$	$y = 1 + 12$
$y = 4$	$y = 13$
Lower	upper

③ $A = \int_{1}^{5} [(-15 + 8x + 3x^2) - (-10x + 6x^2)]\,dx$

$= \int_{1}^{5} [-3x^2 + 18x - 15]\,dx$

$= [-x^3 + 9x^2 - 15x] \Big|_{1}^{5}$

$= [-(5)^3 + 9(5)^2 - 15(5)] - [-(1)^3 + 9(1)^2 - 15(1)]$

$= [-125 + 9(25) - 75] - [-(1) + 9(1) - 15]$

$= [-125 + 225 - 75] - [-1 + 9 - 15]$

$= [25] - [-7]$

$A = 32 \text{ un}^2$

#11)

area between	$y = 2 - 12x + 3x^2$
	$y = 2$

① cross? Yes (0,4)

$$2 = 2 - 12x + 3x^2$$
$$0 = 3x^2 - 12x$$
$$0 = 3x(x-4)$$
$$0 = 3x \quad \} \quad 0 = x-4$$
$$0 = x \quad \} \quad 4 = x$$
$$(0,4)$$

upper/Lower
$1 \in (0,4)$

$2 = y$	$y = 2 - 12x + 3x^2$
	$y = 2 - 12(1) + 3(1)^2$
	$y = 2 - 12 + 3(1)$
	$y = -10 + 3$
	$y = -7$
upper	Lower

③ $A = \int_0^4 [(2) - (2 - 12x + 3x^2)]\,dx$

$= \int_0^4 [12x - 3x^2]\,dx$

$= [6x^2 - x^3] \Big|_0^4$

$= [6(4)^2 - (4)^3] - [6(0)^2 - (0)^3]$

$= [6(16) - 64] - [0]$

$= [96 - 64]$

$A = 32 \text{ un}^2$

#12)

area between	$y = x^2$
	$y = 1$

① cross? Yes (-1,1)

$$x^2 = 1$$
$$x = \pm 1$$
$$(-1, 1)$$

② upper/Lower
$0 \in (-1,1)$

$y = x^2$	$y = 1$
$y = (0)^2$	
$y = 0$	
Lower	upper

③ $A = \int_{-1}^{1} [(1) - (x^2)]\,dx$

$= [x - \frac{1}{3}x^3] \Big|_{-1}^{1}$

$= [(1) - \frac{1}{3}(1)^3] - [(-1) - \frac{1}{3}(-1)^3]$

$= [1 - \frac{1}{3}(1)] - [-1 - \frac{1}{3}(-1)]$

$= [1 - \frac{1}{3}] - [-1 + \frac{1}{3}]$

$= 1 - \frac{1}{3} + 1 - \frac{1}{3}$

$= 2 - \frac{2}{3}$

$= \frac{6}{3} - \frac{2}{3}$

$A = \frac{4}{3} \text{ un}^2$

#13)

area between	$y = \cos(x)$
	$y = x^2$

Round points of intersection to 4 decimal places and final answer to 2 decimal places.

(Use your calculator to determine where the graphs intersect. Also use your calculator to determine which is the upper and which is the lower curve.)

① cross? Yes, (−0.8241, 0.8241)

$\cos(x) = x^2$

USE CALCULATOR

$Y_1 = x^2$ MODE: RADIAN
$Y_2 = \cos(x)$ WINDOW: x: [−2π, 2π]
y: [−1, 1]

2nd CALC
5: INTERSECT

$x \approx -0.8241, 0.8241$

② upper: $y = \cos(x)$
Lower: $y = x^2$

③ $A \approx \int_{-0.8241}^{0.8241} [\cos(x) - x^2]\,dx$

$\approx \left[\sin(x) - \frac{1}{3}x^3\right]\Big|_{-0.8241}^{0.8241}$

$\approx \left[\sin(0.8241) - \frac{1}{3}(0.8241)^3\right] - \left[\sin(-0.8241) - \frac{1}{3}(-0.8241)^3\right]$

$A \approx 1.09$ un²

Since we have to round, just put whole thing into calculator.

Bieber Fever

#14) Justin Bieber's Twitter followers are increasing at a rate of $y = 22e^{0.02t}$ million new followers per year, where t is the number of years after 2014. George's Twitter followers are increasing at a rate of $y = 2t + 1$ million new followers per year. Find how many more new Twitter followers the Beeb has compared to George from 2014 to 2017.

t = years after 2014
F = more Twitter Follows (millions)

(Use your calculator to determine where the graphs intersect. Find which is upper and lower by hand.)

① cross?
$22e^{0.02t} = 2t+1$

USE CALCULATOR

$Y_1 = 22e^{0.02t}$
$Y_2 = 2t+1$ WINDOW: x: [−20, 20]
y: [0, 50]

2nd CALC
5: INTERSECT

$x = 14.08 \notin [0, 3]$

② upper / Lower
$1 \in [0, 3]$

$y = 22e^{0.02t}$	$y = 2t+1$
$y = 22e^{0.02(1)}$	$y = 2(1)+1$
$y = 22e^{0.02}$	$y = 3+1$
$y \approx 22.4$	$y = 4$
upper	Lower

③ $F = \int_0^3 \left[(22e^{0.02t}) - (2t+1)\right] dt$

$= \int_0^3 [22e^{0.02t} - 2t - 1]\,dt$

$= \left[22(50)e^{0.02t} - t^2 - t\right]\Big|_0^3$

$= \left[1100e^{0.02(3)} - (3)^2 - (3)\right] - \left[1100e^{0.02(0)} - (0)^2 - (0)\right]$

$= \left[1100e^{0.06} - 9 - 3\right] - \left[1100e^0 - 0\right]$

$= 1100e^{0.06} - 12 - 1100$

$= 1100e^{0.06} - 1112$

$F \approx 56.020201$ million

Bieber will have about 56,020,201 more followers than George from 2014 to 2017.

The Hoff

#15) The number of girls David Hasselholf can get to jump in his car before starring in Knight Rider was growing at a rate of $y = \frac{1}{x}$ girls per week, where $x = 1$ corresponds to the first week he starred in Knight Rider. Once starring as Michael Knight in The Knight Rider the number of girls David Hasselholf could get to jump in his car grew at a rate of $y = x^2$ girls per week. Find how many more girls jumped in his car because he was in Knight Rider (verses him never being in the show) for the first 5 weeks of the show. Round to the nearest girl.

① Cross? Not in (1,6]

$\frac{1}{x} = x^2$

$1 = x^3$

$1 = x$

$1 \notin (1,6]$

x = weeks starring in show

G = more Girls jumping in car

② Upper/Lower

$2 \in (1,6]$

$y = \frac{1}{x}$	$y = x^2$
	$y = (2)^2$
$y = \frac{1}{2}$	$y = 4$
Lower	Upper

③ $G = \int_1^6 \left[x^2 - \frac{1}{x}\right] dx$

$= \left[\frac{1}{3}x^3 - \ln|x|\right]\Big|_1^6$

$= \left[\frac{1}{3}(6)^3 - \ln|6|\right] - \left[\frac{1}{3}(1)^3 - \ln|1|\right]$

$= \left[\frac{1}{3}(216) - \ln 6\right] - \left[\frac{1}{3}(1) - 0\right]$

$= \frac{216}{3} - \ln 6 - \frac{1}{3}$

$= \frac{215}{3} - \ln 6$

$G \approx 70$

For the first 5 weeks the Hoff would have about 70 more girls jump in his car because of Knight RIDER

Somebody George Used to Know

#16) After an unfortunate accident with a retired sports celebrity, a knife and his own skull, George's memory isn't what it once was. The number of people George used to know is growing at a rate of $y = x^3 + x^2 + 5$ people per day, where $x = 0$ corresponds to today. The number of people that George will know is growing at a rate of $y = 4x$ people per day. Find how many more people George used to know verses he will know 7 days from now.

(Use your calculator to determine where the graphs intersect. Also use your calculator to determine which is the upper and which is the lower curve.)

① Cross? Not

$x^3 + x^2 + 5 = 4x$

$x^3 + x^2 - 4x + 5 = 0$

Prime

CALCULATOR

$x \approx -2.9 \notin [0,7]$

x = days

F = more people used to know vs know

② Upper: $y = x^3 + x^2 + 5$

Lower: $y = 4x$

③ $F = \int_0^7 \left[(x^3 + x^2 + 5) - (4x)\right] dx$

$= \int_0^7 \left[x^3 + x^2 - 4x + 5\right] dx$

$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 2x^2 + 5x\right]\Big|_0^7$

$= \left[\frac{1}{4}(7)^4 + \frac{1}{3}(7)^3 - 2(7)^2 + 5(7)\right] - \left[\frac{1}{4}(0)^4 + \frac{1}{3}(0)^3 - 2(0)^2 + 5(0)\right]$

$= \left[\frac{1}{4}(2401) + \frac{1}{3}(343) - 2(49) + 35\right] - [0]$

$= \left[\frac{2401}{4} + \frac{343}{3} - 98 + 35\right]$

$= \frac{2401}{4} + \frac{343}{3} - 63$

$= \frac{7203}{12} + \frac{1372}{12} - \frac{756}{12}$

$= \frac{7819}{12}$

$F \approx 652$ people

In the next 7 days George used to know about 652 more people than he will know.

Answers

#1) 11/6 un^2

#2) 2 un^2

#3) $(e^2 - e - \ln 2)un^2 \approx 3.97763\ un^2$

#4) $\left(2e + \frac{2}{e} - 4\right)un^2 \approx 2.1723\ un^2$

#5) 9 un^2

#6) 15 un^2

#7) 0.63 un^2

#8) 17 un^2

#9) 4 un^2

#10) 32 un^2

#11) 32 un^2

#12) 4/3 un^2

#13) 1.09 un^2

#14) Justin will have about 56,020,200 more followers than George.

#15) For the first five weeks of Knight Rider, David would have about 70 more girls jump in his car because he was on the show.

#16) In the next seven days George used to know about 652 more people than he will know.