Basic Integration

8.3 – Area Between Curves

How to complete the square…

#1) $x^2 + 10x - 12 = 0$

$(x^2 + 10x + 25) - 12 - 25 = 0$

$(x+5)^2 - 27 = 0$

$(x+5)^2 = 27$

$x + 5 = \pm\sqrt{27}$

$x = -5 \pm \sqrt{27}$ → $x = -10.2, 0.2$

#2) $x^2 - 5x + 10 = 0$

$\left(x^2 - 5x + \left(\frac{5}{2}\right)^2\right) + 10 - \frac{25}{4} = 0$

$\left(x - \frac{5}{2}\right)^2 + \frac{40}{4} - \frac{25}{4} = 0$

$\left(x - \frac{5}{2}\right)^2 + \frac{15}{4} = 0$

$\left(x - \frac{5}{2}\right)^2 = \frac{-15}{4}$ → $x - \frac{5}{2} = \pm\sqrt{\frac{-15}{4}}$

$x = \frac{5}{2} \pm i\sqrt{\frac{15}{4}}$

$x = dne$

Area Between Curves

The area between two continuous curves can be written as a single integral:

$$\text{Area between } f \text{ and } g \text{ on } [a, b] = \int_a^b [f(x) - g(x)]dx$$

How to find the area between two functions that may or may not cross

Step 1: Do they cross?

If two functions cross, we want to change the original interval into subintervals.

To do so, find out where the curves cross by setting them equal to each other and solving for x.

Only use x-values that are in the given interval.

Use this new value to break apart the interval into subintervals.

Step 2: Upper/Lower

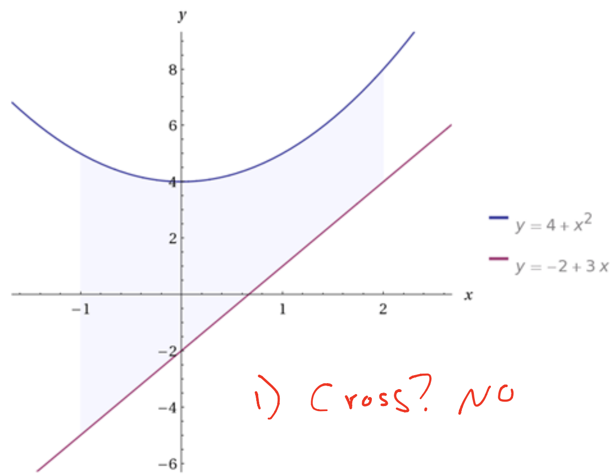
You now have to figure out which is the upper curve and which is the lower curve. Pick a test point to do so. BE CAREFUL. You must pick a test point from each subinterval, **but you cannot use the number shared by both intervals.**

Step 3: Integrate

Integrate (the difference of the upper curve and lower curve) on the first interval + (the difference of the upper curve and lower curve) on the second interval.

Ex A: Find the area between the curves that may or may not cross.

#1) area between $y = 4 + x^2$ and $y = -2 + 3x$, domain $-1 \le x \le 2$



1) Cross? NO

2) upper / lower

$0 \in [-1, 2]$

$y = 4 + x^2$	$y = -2 + 3x$
$y = 4 + (0)^2$	$y = -2 + 3(0)$
$y = 4$	$y = -2$
Upper	Lower

3) $A = \int_a^b [(\text{upper}) - (\text{lower})]dx$

$A = \int_{-1}^{2} [(4 + x^2) - (-2 + 3x)]dx$

$= \int_{-1}^{2} (x^2 - 3x + 6)dx$

$= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 6x\right)\Big|_{-1}^{2}$

$= \left[\frac{1}{3}(2)^3 - \frac{3}{2}(2)^2 + 6(2)\right] - \left[\frac{1}{3}(-1)^3 - \frac{3}{2}(-1)^2 + 6(-1)\right]$

$= \left[\frac{1}{3}(8) - \frac{3}{2}(4) + 12\right] - \left[\frac{1}{3}(-1) - \frac{3}{2}(1) - 6\right]$

$= \left[\frac{8}{3} + \frac{-12}{2} + 12\right] - \left[-\frac{1}{3} - \frac{3}{2} - 6\right]$

$= \frac{8}{3} - 6 + 12 + \frac{1}{3} + \frac{3}{2} + 6$

$= \frac{9}{3} + 12 + \frac{3}{2}$

$= 3 + 12 + \frac{3}{2}$

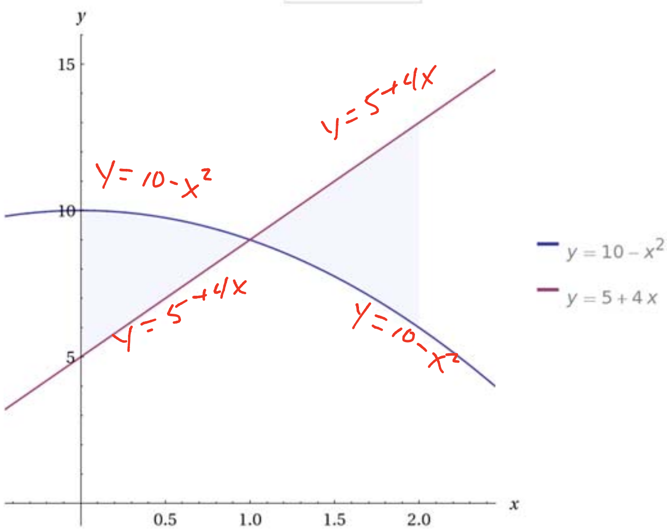
$= 15 + 1.5$

$A = 16.5 \text{ un}^2$

Basic Integration
8.3 – Area Between Curves

#2)

area between	$y = 10 - x^2$ $y = 5 + 4x$	domain	$0 \le x \le 2$



1) Cross? Yes @ 1

$10 - x^2 = 5 + 4x$

$0 = x^2 + 4x - 5$

$0 = (x+5)(x-1)$

$0 = x + 5$; $0 = x - 1$

$-5 = x$; $1 = x$

$-5 \notin [0,2]$; $1 \in [0,2]$

$[0,1)$
- $y = 10 - x^2$ upper
- $y = 5 + 4x$ Lower

$(1,2]$
- $y = 5 + 4x$ upper
- $y = 10 - x^2$ Lower

$$A = \int_0^1 [(10 - x^2) - (5 + 4x)]dx + \int_1^2 [(5+4x) - (10 - x^2)]dx$$

$$= \int_0^1 [-x^2 - 4x + 5]dx + \int_1^2 [x^2 + 4x - 5]dx$$

$$= \left[-\tfrac{1}{3}x^3 - 2x^2 + 5x\right]\Big|_0^1 + \left[\tfrac{1}{3}x^3 + 2x^2 - 5x\right]\Big|_1^2$$

$$= \left[-\tfrac{1}{3}(1)^3 - 2(1)^2 + 5(1)\right] - \left[-\tfrac{1}{3}(0)^3 - 2(0)^2 + 5(0)\right] + \left[\tfrac{1}{3}(2)^3 + 2(2)^2 - 5(2)\right] - \left[\tfrac{1}{3}(1)^3 + 2(1)^2 - 5(1)\right]$$

$$= \left[-\tfrac{1}{3}(1) - 2(1) + 5\right] - [0] + \left[\tfrac{1}{3}(8) + 2(4) - 10\right] - \left[\tfrac{1}{3}(1) + 2(1) - 5\right]$$

$$= \left[-\tfrac{1}{3} - 2 + 5\right] + \left[\tfrac{8}{3} + 8 - 10\right] - \left[\tfrac{1}{3} + 2 - 5\right]$$

$$= -\tfrac{1}{3} + 3 + \tfrac{8}{3} - 2 - \tfrac{1}{3} + 3$$

$$= 4 + \tfrac{6}{3}$$

$$= 4 + 2$$

$$= 6 \text{ un}^2$$

#3)

area between	$y = -3 + x^2$ $y = 3 - x^2$	domain	$-1 \le x \le 1$

1) Cross? Not in $[-1,1]$

$-3 + x^2 = 3 - x^2$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

$\pm 3 \notin [-1,1]$

2) Upper/Lower

$0 \in [-1,1]$

$y = -3 + x^2$	$y = 3 - x^2$
$y = -3 + (0)^2$	$y = 3 - (0)$
$y = -3$	$y = 3$
Lower	upper

3) $$A = \int_{-1}^{1} [(3 - x^2) - (-3 + x^2)]dx$$

$$= \int_{-1}^{1} [-2x^2 + 6]dx$$

$$= \left[-\tfrac{2}{3}x^3 + 6x\right]\Big|_{-1}^{1}$$

$$= \left[-\tfrac{2}{3}(1)^3 + 6(1)\right] - \left[-\tfrac{2}{3}(-1)^3 + 6(-1)\right]$$

$$= \left[-\tfrac{2}{3}(1) + 6\right] - \left[-\tfrac{2}{3}(-1) - 6\right]$$

$$= \left[-\tfrac{2}{3} + 6\right] - \left[\tfrac{2}{3} - 6\right]$$

$$= -\tfrac{2}{3} - \tfrac{2}{3} + 6 + 6$$

$$= -\tfrac{4}{3} + 12$$

$$= -\tfrac{4}{3} + \tfrac{36}{3}$$

$$A = \tfrac{32}{3} \text{ un}^2$$

#4)

area between	$y=2+x^2$	domain	$1\le x\le 3$
	$y=2+2x$		

1) Cross? Yes @ 1

$2+x^2=2+2x$

$x^2-2x=0$

$x(x-2)=0$

$x=0$ or $x-2=0$, $x=2$

$0\notin[1,3]$ $2\in[1,3]$

(2) Upper/Lower

$1\in[1,2)$

$y=2+x^2$	$y=2+2x$
$y=2+(1)^2$	$y=2+2(1)$
$y=2+1$	$y=2+2$
$y=3$	$y=4$
Lower	Upper

Upper/Lower

$3\in(2,3]$

$y=2+x^2$	$y=2+2x$
$y=2+(3)^2$	$y=2+2(3)$
$y=2+9$	$y=2+6$
$y=11$	$y=8$
Upper	Lower

(3) $A=\int_1^2[(2+2x)-(2+x^2)]dx + \int_2^3[(2+x^2)-(2+2x)]dx$

$=\int_1^2[-x^2+2x]dx + \int_2^3[x^2-2x]dx$

$=\left[-\frac{1}{3}x^3+x^2\right]_1^2 + \left[\frac{1}{3}x^3-x^2\right]_2^3$

$=\left[-\frac{1}{3}(2)^3+(2)^2\right]-\left[-\frac{1}{3}(1)^3+(1)^2\right]+\left[\frac{1}{3}(3)^3-(3)^2\right]-\left[\frac{1}{3}(2)^3-(2)^2\right]$

$=\left[-\frac{1}{3}(8)+4\right]-\left[-\frac{1}{3}(1)+1\right]+\left[\frac{1}{3}(27)-9\right]-\left[\frac{1}{3}(8)-4\right]$

$=\left[-\frac{8}{3}+4\right]-\left[-\frac{1}{3}+1\right]+\left[9-9\right]-\left[\frac{8}{3}-4\right]$

$=-\frac{8}{3}+\frac{1}{3}+4-1 + [0]-\frac{8}{3}+4$

$=-\frac{7}{3}+3 + -\frac{8}{3}+4$

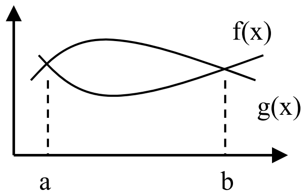
$=-\frac{15}{3}+7$

$=-5+7$

$A=2\ un^2$

Areas Bounded by Curves

Some problems ask for the area bounded by two curves, without giving the starting and ending x-values. In such problems the curves completely enclose an area, and the *x*-values for the upper and lower limits of integration are found by setting the functions equal to each other and solving.



minus

Ex B: Finding an Area Bounded by Curves

#1)

area between	$y=-1+x^2$
	$y=1-x^2$

1) Cross? Yes @ -1 and 1

$-1+x^2=1-x^2$

$2x^2-2=0$

$2(x^2-1)=0$

$2(x-1)(x+1)=0$

$x-1=0$, $x+1=0$

$x=1$, $x=-1$

2) Upper/Lower

$0\in(-1,1)$

$y=-1+x^2$	$y=1-x^2$
$y=-1+(0)^2$	$y=1-(0)^2$
$y=-1$	$y=1$
Lower	Upper

(3) $A=\int_{-1}^{1}[(1-x^2)-(-1+x^2)]dx$

$=\int_{-1}^{1}[-2x^2+2]dx$

$=\left[-\frac{2}{3}x^3+2x\right]_{-1}^{1}$

$=\left[-\frac{2}{3}(1)^3+2(1)\right]-\left[-\frac{2}{3}(-1)^3+2(-1)\right]$

$=\left[-\frac{2}{3}(1)+(2)\right]-\left[-\frac{2}{3}(-1)-2\right]$

$=\left[-\frac{2}{3}+2\right]-\left[\frac{2}{3}-2\right]$

$=-\frac{2}{3}-\frac{2}{3}+2+2$

$=-\frac{4}{3}+4$

$=-\frac{4}{3}+\frac{12}{3}$

$A=\frac{8}{3}\ un^2$

FYI: If the two curves represent rates (one unit per another unit), then the area between the curves gives the total accumulation at the upper rate minus the lower rate.

Ex D: Finding Sales from Extra Advertising

Knever Kdull

#1) Inspired by the *30 for 30* film *June 17, 1994*, George starts selling knives. George's Knever Kdull Company expects to sell knives at the rate of $y = e^{0.2t}$ thousand knives per month, where t is the number of months since they became available. However, with additional advertising using sports celebrity OJ Simpson, they should sell at the rate of $y = e^{0.1t}$ thousand knives per month. How many additional sales would result from the celebrity endorsement during the first year?

t = month
AS = Add'n Sales in thousands

Since t is in months and we are looking for sales during the 1st year, the interval is [0, 12]

① Cross? Yes @ 0

$e^{0.2t} = e^{0.1t}$

$\ln(e^{0.2t}) = \ln(e^{0.1t})$

$0.2t = 0.1t$

$0.1t = 0$

$t = 0$

$0 \in [0, 12]$

② upper/Lower

$1 \in (0, 12]$

upper	Lower
$y = e^{0.2t}$	$y = e^{0.1t}$
$y = e^{0.2(1)}$	$y = e^{0.1(1)}$
$y = e^{0.2}$	$y = e^{0.1}$
$y = e^{0.2}$	$y = e^{0.1}$

This means OJ will dereage sales!

③ $AS = \int_0^{12} [(e^{0.2t}) - (e^{0.1t})]\,dt$

$= \left[5e^{0.2t} - 10e^{0.1t}\right]\Big|_0^{12}$

$= \left[5e^{0.2(12)} - 10e^{0.1(12)}\right] - \left[5e^{0.2(0)} - 10e^{0.1(0)}\right]$

$= \left[5e^{2.4} - 10e^{1.2}\right] - \left[5e^{0} - 10e^{0}\right]$

$= 5e^{2.4} - 10e^{1.2} - 5 + 10$

$= 5e^{2.4} - 10e^{1.2} + 5$

$AS \approx 26.915$ thousand knives

OJ Simpson's endorsement will result in 26,915 fewer knife sales during the 1st year.