Basic Integration
9.1B – U Substitution

A: Integrate each definite integral with a u-substitution.

#1) $\int_0^2 (x^2-1)^5 2x\,dx$

$$\boxed{\begin{aligned} u &= x^2-1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{du}{2x} &= dx \end{aligned}}$$

$$= \int_{x=0}^{x=2} u^5 \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$$
$$= \int_{x=0}^{x=2} u^5\,du$$
$$= \frac{1}{6}u^6 \Big|_{x=0}^{x=2}$$
$$= \frac{1}{6}(x^2-1)^6 \Big|_0^2$$
$$= \frac{1}{6}\left\{[(2)^2-1]^6 - [(0)^2-1]^6\right\}$$
$$= \frac{1}{6}\left\{[4-1]^6 - [0-1]^6\right\}$$
$$= \frac{1}{6}\left\{[3]^6 - [-1]^6\right\}$$
$$= \frac{1}{6}[729-1]$$
$$= \frac{728}{6}$$
$$= \frac{364}{3}$$

#2) $\int_0^1 e^{x^2} 2x\,dx$

$$\boxed{\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{du}{2x} &= dx \end{aligned}}$$

$$= \int_{x=0}^{x=1} e^u \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$$
$$= \int_{x=0}^{x=1} e^u\,du$$
$$= e^u \Big|_{x=0}^{x=1}$$
$$= e^{x^2} \Big|_0^1$$
$$= \left[e^{(1)^2}\right] - \left[e^{(0)^2}\right]$$
$$= e^1 - e^0$$
$$= e - 1$$

#3) $\int_{-3}^{3} \frac{2x}{x^2-1}\,dx$

$$\boxed{\begin{aligned} u &= x^2-1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{du}{2x} &= dx \end{aligned}}$$

$$= \int_{x=-3}^{x=3} \frac{\cancel{2x}}{u}\left(\frac{du}{\cancel{2x}}\right)$$
$$= \int_{x=-3}^{x=3} \frac{1}{u}\,du$$
$$= \ln|u| \Big|_{x=-3}^{x=3}$$
$$= \ln|x^2-1| \Big|_{-3}^{3}$$
$$= \left[\ln|(3)^2-1|\right] - \left[\ln|(-3)^2-1|\right]$$
$$= \ln|9-1| - \ln|9-1|$$
$$= \ln|8| - \ln|8|$$
= ☺

#4) $\int_{-1}^{1} (x^2-1)^5 x\,dx$

$$\boxed{\begin{aligned} u &= x^2-1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{du}{2x} &= dx \end{aligned}}$$

$$= \int_{x=-1}^{x=1} u^5 \cancel{x}\left(\frac{du}{2\cancel{x}}\right)$$
$$= \frac{1}{2}\int_{x=-1}^{x=1} u^5\,du$$
$$= \frac{1}{12}u^6 \Big|_{x=-1}^{x=1}$$
$$= \frac{1}{12}(x^2-1)^6 \Big|_{-1}^{1}$$
$$= \frac{1}{12}\left[[(1)^2-1]^6 - [(-1)^2-1]^6\right]$$
$$= \frac{1}{12}\left[[1-1]^6 - [1-1]^6\right]$$
$$= \frac{1}{12}\left[[0]^6 - [0]^6\right]$$
$$= \frac{1}{12}[0]$$
= ☺

#5) $\int_1^2 e^{x^3}3x^2\,dx$

$u=x^3$

$\frac{du}{dx}=3x^2$

$du=3x^2\,dx$

$\frac{du}{3x^2}=dx$

$$=\int_{x=1}^{x=2} e^u\,\cancel{3x^2}\left(\frac{du}{\cancel{3x^2}}\right)$$

$$=\int_{x=1}^{x=2} e^u\,du$$

$$=e^u\Big|_{x=1}^{x=2}$$

$$=e^{x^3}\Big|_{x=1}^{x=2}$$

$$=\left[e^{(2)^3}\right]-\left[e^{(1)^3}\right]$$

$$=e^8-e$$

#6) $\int_{-2}^2 \frac{x^2}{x^3+2}\,dx$

$u=x^3+2$

$\frac{du}{dx}=3x^2$

$du=3x^2\,dx$

$\frac{du}{3x^2}=dx$

$$=\int_{x=-2}^{x=2}\frac{\cancel{x^2}}{u}\left(\frac{du}{3\cancel{x^2}}\right)$$

$$=\frac{1}{3}\int_{x=-2}^{x=2}\frac{1}{u}\,du$$

$$=\frac{1}{3}\ln|u|\Big|_{x=-2}^{x=2}$$

$$=\frac{1}{3}\ln|x^3+2|\Big|_{-2}^{2}$$

$$=\left[\frac{1}{3}\ln|(2)^3+2|\right]-\left[\frac{1}{3}\ln|(-2)^3+2|\right]$$

$$=\left[\frac{1}{3}\ln|8+2|\right]-\left[\frac{1}{3}\ln|-8+2|\right]$$

$$=\frac{1}{3}\ln|10|-\frac{1}{3}\ln|-6|$$

$$=\frac{1}{3}\left[\ln 10-\ln 6\right]$$

$$=\frac{1}{3}\ln\left(\frac{5}{3}\right)$$

Bathroom Tissue

#7) Adding to his line of products for *The Slightly Used Company,* George starts selling bathroom tissue. *Slightly Used's* marginal cost function is $MC(x)=\frac{1}{4x+2}$ and its fixed costs are \$4. Find the cost function.

$(0,4)$

$u=4x+2$

$\frac{du}{dx}=4$

$du=4\,dx$

$\frac{du}{4}=dx$

$$C(x)=\int MC(x)\,dx$$

$$=\int\frac{1}{4x+2}\,dx$$

$$=\int\frac{1}{u}\left(\frac{du}{4}\right)$$

$$=\frac{1}{4}\int\frac{1}{u}\,du$$

$$=\frac{1}{4}\ln|u|+C$$

$$C(x)=\frac{1}{4}\ln|4x+2|+C$$

$$4=\frac{1}{4}\ln|4(0)+2|+C$$

$$4=\frac{1}{4}\ln|2|+C$$

$$16=\ln 2+C$$

$$16-\ln 2=C$$

$$C(x)=\frac{1}{4}\ln|4x+2|+16-\ln 2$$

Pluckable Hairs

#8) The number of pluckable hairs on George's ears is expected to be $P(x)=x(x^2+4)^{-1/2}$ hairs after x months. Find the average number of pluckable hairs between month $x=0$ and month $x=8$.

$$\text{Average Pluckable Hairs}=\frac{1}{8-0}\int_0^8 x(x^2+4)^{-\frac{1}{2}}\,dx$$

$u=x^2+4$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$\frac{du}{2x}=dx$

$$=\frac{1}{8}\int_{x=0}^{x=8}\cancel{x}\,u^{-\frac{1}{2}}\left(\frac{du}{2\cancel{x}}\right)$$

$$=\frac{1}{8}\cdot\frac{1}{2}\int_{x=0}^{x=8}u^{-\frac{1}{2}}\,du$$

$$=\frac{1}{8}u^{\frac{1}{2}}\Big|_{x=0}^{x=8}$$

$$=\frac{1}{8}\sqrt{x^2+4}\Big|_0^8$$

$$=\frac{1}{8}\left[\sqrt{(8)^2+4}\right]-\frac{1}{8}\left[\sqrt{(0)^2+4}\right]$$

$$=\frac{1}{8}\sqrt{64+4}-\frac{1}{8}\sqrt{0+4}$$

$$=\frac{1}{8}\sqrt{68}-\frac{1}{8}\sqrt{4}$$

$$=\frac{1}{8}\sqrt{68}-\frac{1}{4}$$

Average Pluckable Hairs $\approx .78$ per month

The average number of pluckable hairs from month 0 to 8 is .78 per month.

Alliteration – The Prequel

#9) George sells sails for snail sized sailboats. His sales of sails for snail sized sailboats during week x are given by $S(x) = \frac{1}{x+4}$ in hundreds. Find the average sales of sails for snail sized sailboats from week $x = 1$ to week $x = 4$. (Don't forget your answer is in hundreds, noob.)

$$AS = \frac{1}{4-1}\int_1^4 \frac{1}{x+4}\,dx$$

$$\boxed{\begin{array}{l} u = x+4 \\ \frac{du}{dx} = 1 \\ du = dx \end{array}}$$

$$= \frac{1}{4}\int_{x=1}^{x=4} \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u|\Big|_{x=1}^{x=4}$$

$$= \frac{1}{4}\ln|x+4|\Big|_1^4$$

$$= \frac{1}{4}\left[\ln|(4)+4| - \ln|(1)+4|\right]$$

$$= \frac{1}{4}\left[\ln 8 - \ln 5\right] = .12 \text{ hundred}$$

$$= \frac{1}{4}\ln\frac{8}{5} \text{ hundred} = 12$$

The average sales of sails was 12 from week 1 to 4.

Alliteration

#10) An experimental therapy lowers a patient's patience for patterns at the rate of $t\sqrt{36-t^2}$ units per day, where t is the number of days since the therapy was administered (for the first six days). Find the total change in a patient's patience for patterns during the first 3 days.

$$PPP = \int_0^3 t\sqrt{36-t^2}\,dt$$

$$\boxed{\begin{array}{l} u = 36-t^2 \\ \frac{du}{dt} = -2t \\ du = -2t\,dt \\ \frac{du}{-2t} = dt \end{array}}$$

$$= \int_{t=0}^{t=3} \cancel{t}\sqrt{u}\left(\frac{du}{-2\cancel{t}}\right)$$

$$= -\frac{1}{2}\int_{t=0}^{t=3} u^{\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_{t=0}^{t=3}$$

$$= -\frac{1}{3}\left(\sqrt{36-t^2}\right)^3\Big|_0^3$$

$$= -\frac{1}{3}\left[\left(\sqrt{36-(3)^2}\right)^3 - \left(\sqrt{36-(0)^2}\right)^3\right]$$

$$= -\frac{1}{3}\left[\left(\sqrt{36-9}\right)^3 - \left(\sqrt{36}\right)^3\right]$$

$$= -\frac{1}{3}\left[\left(\sqrt{27}\right)^3 - 6^3\right]$$

$$= -\frac{1}{3}\left[\left(\sqrt{27}\right)^3 - 216\right]$$

$$\approx 25$$

A patient's patience for patterns lowers by 25 units during the first 3 days.

Condiments

#11) George has developed a new business model for making money in the restaurant business – give away the food for free, but charge for the condiments. He is selling condiments at the rate of $100e^{-x}$ per week after *x* weeks. How many condiments will be sold during the first 8 weeks?

$$C = \int_0^8 100e^{-x}\,dx$$

$u = -x$

$\frac{du}{dx} = -1$

$du = -dx$

$-du = dx$

$$= \int_{x=0}^{x=8} 100e^{u}(-du)$$

$$= -100e^{u}\Big|_{x=0}^{x=8}$$

$$= -100e^{-x}\Big|_0^8$$

$$= \left[-100e^{-(8)}\right] - \left[-100e^{-(0)}\right]$$

$$= \frac{-100}{e^8} + 100e^{(0)}$$

$$= \frac{-100}{e^8} + 100$$

$C \approx 100$ condiments sold

George will sell 100 condiments during the first week.

Discharging Pits

#12) George's armpits are discharging pollution into the air at the rate of $r(t)$ liters per year given by $r(t) = \frac{1}{t+1}$ where *t* is the number of years since George washed. Find the total amount of pollution discharged during the first 3 years of not washing.

$$\text{Total Pollution} = \int_0^3 \frac{1}{t+1}\,dt$$

$u = t+1$

$\frac{du}{dt} = 1$

$du = dt$

$$= \int_{t=0}^{t=3} \frac{1}{u}\,du$$

$$= \ln|u|\Big|_{t=0}^{t=3}$$

$$= \ln|t+1|\Big|_0^3$$

$$= \left[\ln|(3)+1|\right] - \left[\ln|(0)+1|\right]$$

$$= \ln 4 - \ln 1$$

$$= \ln 4 - 0$$

$$= \ln 4$$

$$\approx 1.4 \text{ liters}$$

George's armpits have sent 1.4 liters of pollution into the air during the first 3 years.