# Basic Integration
## 9.1 – U Substitution

**Substitution Method**

Using differential notation, we can state three very useful integration formulas.

(A) $\int u^n du = \frac{1}{n+1}u^{n+1} + C \qquad n \neq -1$

(B) $\int e^u du = e^u + C$

(C) $\int \frac{1}{u} du = \ln|u| + C$

**Why Integration By Substitution?**
Many functions cannot be integrated directly. Some functions can be integrated if a "u-substitution" is first done.

**Differential**
For a differentiable function *f(x)*, the differential *df* is

$$df = f'(x)\,dx$$

**Explanation for Differential**
One of the notations for the derivative of a function *f(x)* is $\frac{df}{dx}$. Although written as a fraction, $\frac{df}{dx}$ was not defined as the quotient of two quantities *df* and *dx*, but as a single object, the *derivative*. We will now define *df* and *dx* separately (they are called differentials) so that their quotient $df \div dx$ is equal to the derivative $\frac{df}{dx}$.

$$f = f$$

$$\frac{df}{dx} = f'$$

$$df = f'\,dx$$

Note that df does NOT mean *d* times *f*. The dx is just the notation that appears at the end of integrals, arising from the Δx in the Riemann sum. The reason for finding the differentials will be made clear shortly.

Baby Step 1 – Solve for the differential $du$

#1) Differentiate each side with respect to x.

#2) Solve equation for du.

#1) $u = 3x^2$

$\frac{du}{dx} = 6x$

$du = 6x\,dx$

#2) $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

#3) $u = \frac{1}{2}e^{x^2}$

$\frac{du}{dx} = xe^{x^2}$

$du = xe^{x^2}dx$

Baby Step 2 – Solve for the differential $dx$

#1) Differentiate each side with respect to x.

#2) Solve equation for dx.

#1) $u = x^3 + 1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{du}{3x^2} = dx$

#2) $u = e^{2t} + 1$

$\frac{du}{dt} = 2e^{2t}$

$du = 2e^{2t}dt$

$\frac{du}{2e^{2t}} = dt$

#3) $u = e^{-5t}$

$\frac{du}{dt} = -5e^{-5t}$

$du = -5e^{-5t}dt$

$\frac{du}{-5e^{-5t}} = dt$

# Basic Integration
## 9.1 – U Substitution

**Substitution Method (Repeated)**

(A) $\int u^n du = \frac{1}{n+1}u^{n+1} + C \qquad n \neq -1$

(B) $\int e^u du = e^u + C$

(C) $\int \frac{1}{u} du = \ln|u| + C$

Baby Step 3 – For each of the following integrals, choose the most appropriate formula: (A), (B), or (C).

#1) $\int e^{5x^2-1} x dx$

B

#2) $\int \frac{x dx}{x^2+1}$

C

#3) $\int (x^4-12)^4 x^3 dx$

A

#4) $\int (x^4-12)^{-1} x^3 dx$

C

Ex A: Integrating by Substitution

#1) $\int (x^2+2)^3 2x\, dx$

$= \int (u)^3 \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$

$= \int u^3 du$

$= \frac{1}{4}u^4 + C$

$= \frac{1}{4}(x^2+2)^4 + C$

$u = x^2+2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

#1) Set $u$ equal to an expression. (Be smart!)

#2) Differentiate both sides of that equation and solve for $dx$.

#3) Substitute in $u$ and substitute for $dx$.

#4) Simplify

#5) Integrate the function.

#6) Finally, replace the $u$ with the expression from step #1.

#2) $\int e^{x^2-4} 2x\, dx$

$= \int e^u \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$

$= \int e^u du$

$= e^u + C$

$= e^{x^2-4} + C$

$u = x^2-4$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

#3) $\int \frac{3x^2}{x^3-7} dx$

$= \int \frac{\cancel{3x^2}}{u}\left(\frac{du}{\cancel{3x^2}}\right)$

$= \int \frac{1}{u} du$

$= \ln|u| + C$

$= \ln|x^3-7| + C$

$u = x^3-7$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

# Basic Integration
## 9.1 – U Substitution

Ex B: Integrate by substitution with extra constants.

#4) $\int (x^2+4)^3 x\,dx$

$= \int u^3 \cancel{x}\left(\frac{du}{2\cancel{x}}\right)$

$= \frac{1}{2}\int u^3\,du$

$= \frac{1}{2}\left(\frac{1}{4}u^4\right)+C$

$= \frac{1}{8}(x^2+4)+C$

$u = x^2+4$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

#1) $\int \sqrt{x^3-3x}\,(x^2-1)dx$

$= \int \sqrt{u}\,(x^2-1)\left(\frac{du}{3x^2-3}\right)$

$= \int \sqrt{u}\,\cancel{(x^2-1)}\,\frac{du}{3\cancel{(x^2-1)}}$

$= \frac{1}{3}\int u^{\frac{1}{2}}du$

$= \frac{1}{3}\left(\frac{2}{3}\right)u^{3/2}+C$

$= \frac{2}{9}\left(\sqrt{x^3-3x}\right)^3+C$

$u = x^3-3x$

$\frac{dy}{dx} = 3x^2-3$

$du = (3x^2-3)dx$

$\frac{du}{3x^2-3} = dx$

#2) $\int e^{\sqrt{x}}x^{-1/2}dx$

$= \int e^u \frac{1}{\cancel{\sqrt{x}}}(2\cancel{\sqrt{x}}\,du)$

$= 2\int e^u du$

$= 2e^u+C$

$= 2e^{\sqrt{x}}+C$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,du = dx$

**Substitution Does NOT Work For All Problems**

Show that substitution cannot be used to integrate $\int e^{x^4}dx$

$= \int e^u\left(\frac{du}{4x^3}\right)$

$u = x^4$

$\frac{dy}{dx} = 4x^3$

$du = 4x^3dx$

$\frac{du}{4x^3} = dx$

Baby Step 4 – Decide Which Integral Can Be Found By Substitution

#1) $\int (x^3+1)^3 x^3 dx$

$\int u^3 x^{\cancel{3}}\left(\frac{du}{3\cancel{x^2}}\right)$

NO

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{du}{3x^2} = dx$

#2) $\int e^{x^2}dx$

$= \int e^u\left(\frac{dy}{2x}\right)$

NO

$u = x^2$

$\frac{dy}{dx} = 2x$

$du = 2x\,dx$

$\frac{dy}{2x} = dx$

### Evaluating Definite Integrals by Substitution

Ex C: Evaluating a Definite Integral by Substitution

#1) $\int_4^5 \frac{dx}{3-x}$

$u = 3-x$

$\frac{du}{dx} = -1$

$du = -dx$

$-du = dx$

$= \int_{x=4}^{x=5} \frac{(-du)}{u}$

$= -\ln|u| \Big|_{x=4}^{x=5}$

$= -\ln|3-x| \Big|_4^5$

$= [-\ln|3-(5)|] - [-\ln|3-(4)|]$

$= -\ln|-2| + \ln|-1|$

$= -\ln 2 + \ln 1$

$= -\ln 2 + 0$

$= -\ln 2$

Ex D: Application

**Marginal Butter**

#1) *I Can't Believe It's Not Butter Inc's* marginal (wink, wink) cost function is $MC(x) = \frac{6x^2}{x^3+1}$ and fixed costs are \$1000. Find the cost function.

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

$C(x) = \int \frac{6x^2}{x^3+1} dx$

$= \int \frac{\cancel{6x^2}^{\,2}}{u} \left(\frac{du}{\cancel{3x^2}}\right)$

$= 2\int \frac{1}{u} du$

$= 2\ln|u| + C$

$C(x) = 2\ln|x^3+1| + C$

$1000 = 2\ln|(0)^3+1| + C$

$1000 = 2\ln|1| + C$

$1000 = 2(0) + C$

$1000 = C$

$C(x) = 2\ln|x^3+1| + 1000$

**George's Chuck**

#2) Frogs are being chucked into a lake by George at the rate of $r(t) = 200te^{t^2}$ per year, where $t$ is the number of years since the Great Frog Shortage of '15. Find the total number of frogs chucked into the lake during the first 2 years.

$u = t^2$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{du}{2t} = dt$

Total Frogs $= \int_0^2 2000t\, e^{t^2} dt$

$= \int_{t=0}^{t=2} \cancel{2000t}^{\,1000}\, e^u \left(\frac{du}{\cancel{2t}}\right)$

$= 1000\int_{t=0}^{t=2} e^u du$

$= 1000e^u \Big|_{t=0}^{t=2}$

$= 1000e^{t^2} \Big|_0^2$

$= [1000e^{(2)^2}] - [1000e^{(0)^2}]$

$= 1000e^4 - 1000e^0$

$= (1000e^4 - 1000)$ Frogs

$\approx 53{,}598$

George chucked about 53,598 frogs into the lake during the first two years

**No Rainbows**

#3) Because of the sheer volume of frogs in the lake, it began to overflow. After $x$ minutes of the lake overflowing, the water level in George's basement is $L(x) = 40x(x^2+9)^{-1/2}$ inches. Find the average depth during the first 4 minutes.

$u = x^2+9$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

Average Depth $= \frac{1}{4-0}\int_0^4 \frac{40x}{\sqrt{x^2+9}} dx$

$= \frac{1}{4}\int_{x=0}^{x=4} \frac{\cancel{40x}^{\,20}}{\sqrt{u}} \left(\frac{du}{\cancel{2x}}\right)$

$= 5\int_{x=0}^{x=4} u^{-\frac{1}{2}} du$

$= 10u^{\frac{1}{2}} \Big|_{x=0}^{x=4}$

$= 10\sqrt{x^2+9} \Big|_0^4$

$= [10\sqrt{(4)^2+9}] - [10\sqrt{(0)^2+9}]$

$= [10\sqrt{16+9}] - [10\sqrt{9}]$

$= 10\sqrt{25} - 10(3)$

$= 10\cdot 5 - 30$

$= 50 - 30$

Average Depth $= 20$ inches

The average depth during the first 4 minutes is 20 inches.