Limits & Continuity
1.3 – One Sided Limits & Graphing

One-Sided Limits

In a one-sided limit, the variable approaches the number from one side only. For example, the limit as x approaches 3 from the left, denoted $x \to 3^-$, means the limit using only x-values to the left of 3, such as 2.9, 2.99, 2.999…. The limit as x approaches 3 from the right, denoted $x \to 3^+$, means the limit using only x-values to the right of 3, such as 3.1, 3.01, 3.001….

Left and Right Limits

$\lim\limits_{x \to c^-} f(x)$ means the limit of $f(x)$ as $x \to c$, but with $x < c$.

$\lim\limits_{x \to c^+} f(x)$ means the limit of $f(x)$ as $x \to c$, but with $x > c$.

Two-sided, Left and Right Limits

$\lim\limits_{x \to c} f(x) = L$ iff both one-sided limits $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist and equal the same number L.

Ex A: Translate into words:

$x \to 3^-$

$x \to -3$

$x \to -3^-$

Ex B: For the piecewise linear function $f(x) = \begin{cases} x+1 & if\ x \le 3 \\ 8-2x & if\ x > 3 \end{cases}$ find the following limits by direct substitution.

a. $\lim\limits_{x \to 3^-} f(x)$ b. $\lim\limits_{x \to 3^+} f(x)$ c. $\lim\limits_{x \to 3} f(x)$

Limits of Functions of Two Variables

Some limits involve two variables, with only one variable approaching a limit.

Ex C: Finding a Limit of a Function of Two Variables

#1) Find $\lim\limits_{h \to 0}(x^2 + xh + h^2)$

#2) Find $\lim\limits_{h \to 0}(3x^2 + 5xh + 1)$

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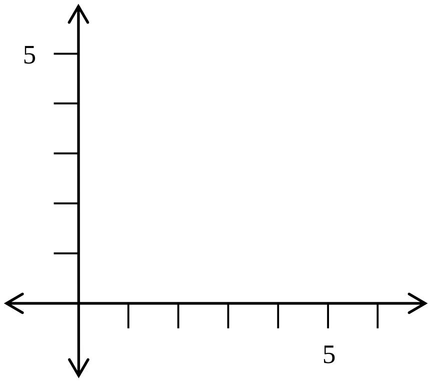
Ex D: Finding One-Sided Limits

For the piecewise linear function $f(x) = \begin{cases} x+1 & if\ x \le 3 \\ 8-2x & if\ x > 3 \end{cases}$ find the following limits by graphing.



a. $\lim\limits_{x \to 3^-} f(x)$

b. $\lim\limits_{x \to 3^+} f(x)$

c. $\lim\limits_{x \to 3} f(x)$

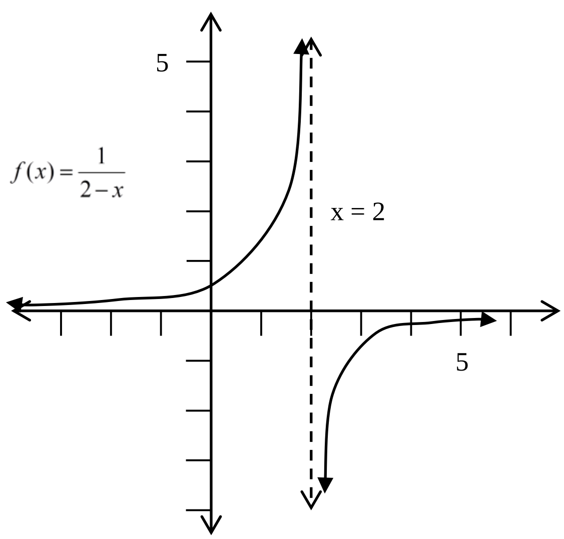
Infinite Limits

We may use the symbols ∞ (infinity) and -∞ (negative infinity) to indicate that the values of a function become arbitrarily large positive or arbitrarily large negative. Dashed lines on a graph, where a function approaches ∞ or -∞, are called vertical asymptotes.

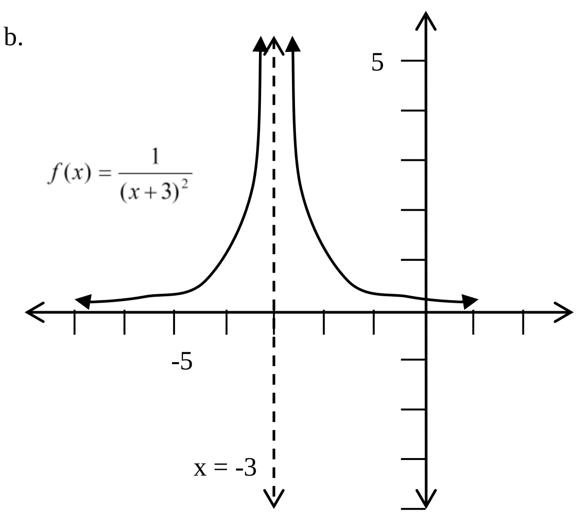
Ex E: Finding Limits Involving $\pm\infty$.

For each function graphed below, use the limit notation with ∞ and -∞ to describe its behavior as *x* approaches the vertical asymptote from the left, from the right, and from both sides.

a.



b.



From the left

From the right

From both sides

From the left

From the right

From the right

Careful. To say that a limit exists is to say that it is a single number. Since ∞ is not a number, if $\lim\limits_{x \to c} f(x) = \infty$, then the limit does not exists (d.n.e.)

A: Translate into words:

#1)

$x \to 10^-$

$x \to -10$

$x \to -10^-$

B: Find each limit by substitution.

#2) $f(x) = \begin{cases} x+6 & if\ x \le 2 \\ 2x-5 & if\ x > 2 \end{cases}$

a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

c. $\lim\limits_{x \to 2} f(x)$

#3) $f(x) = \begin{cases} -3x+6 & if\ x \le 0 \\ 2x+6 & if\ x > 0 \end{cases}$

a. $\lim\limits_{x \to 0^-} f(x)$

b. $\lim\limits_{x \to 0^+} f(x)$

c. $\lim\limits_{x \to 0} f(x)$

B: Find each limit

#4) $f(x) = |x|$

a. $\lim\limits_{x \to 0^-} f(x)$

b. $\lim\limits_{x \to 0^+} f(x)$

c. $\lim\limits_{x \to 0} f(x)$

#5) $f(x) = \frac{|x|}{x}$

a. $\lim\limits_{x \to 0^-} f(x)$

b. $\lim\limits_{x \to 0^+} f(x)$

c. $\lim\limits_{x \to 0} f(x)$

#6) $f(x) = \frac{-x}{|x|}$

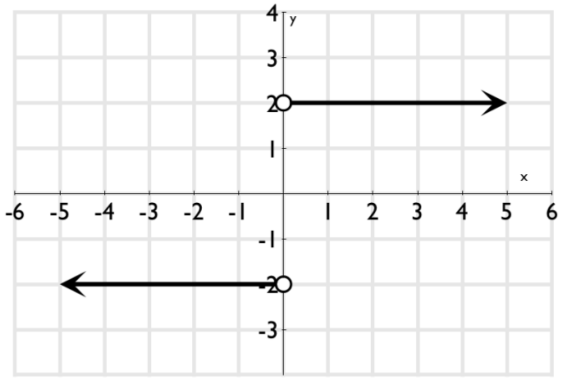
a. $\lim\limits_{x \to 0^-} f(x)$

b. $\lim\limits_{x \to 0^+} f(x)$

c. $\lim\limits_{x \to 0} f(x)$

C: Find each limit.

#7)

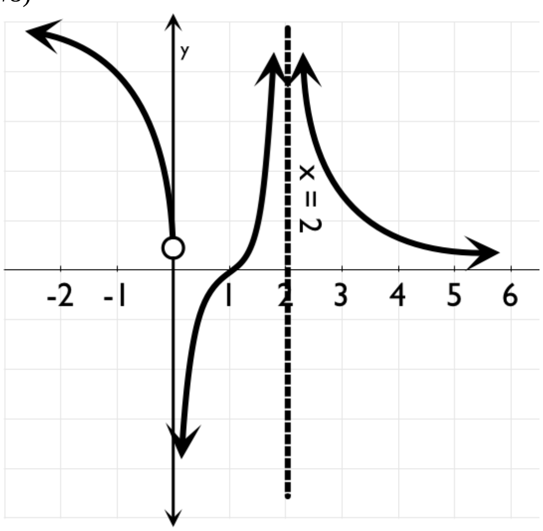


a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

#8)

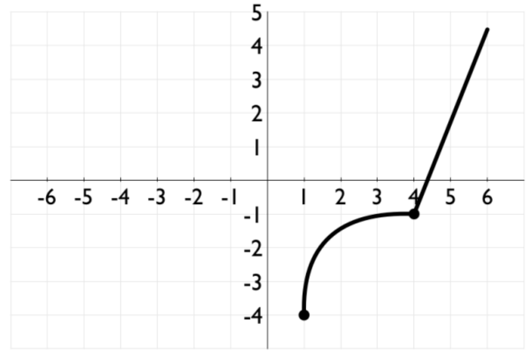


a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

#9)

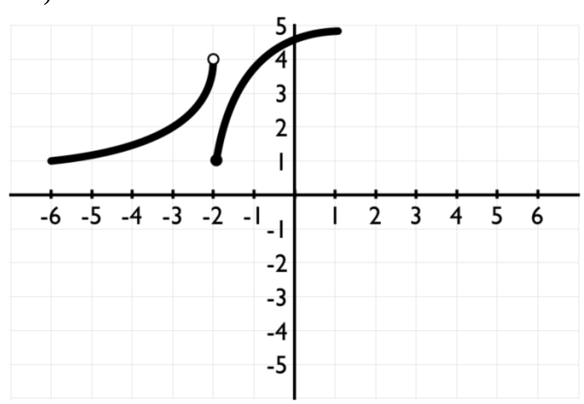


a. $\lim_{x \to 4^-} f(x)$

b. $\lim_{x \to 4^+} f(x)$

c. $\lim_{x \to 4} f(x)$

#10)



a. $\lim_{x \to -2^-} f(x)$

b. $\lim_{x \to -2^+} f(x)$

c. $\lim_{x \to -2} f(x)$

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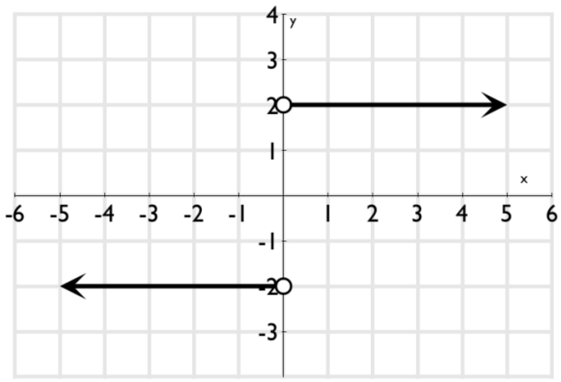
C: Find each limit.

#11)

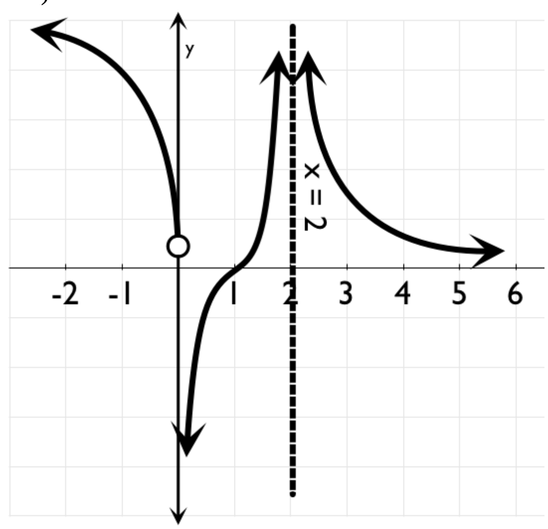


a. $\lim_{x \to 0^-} f(x)$

b. $\lim_{x \to 0^+} f(x)$

c. $\lim_{x \to 0} f(x)$

#12)

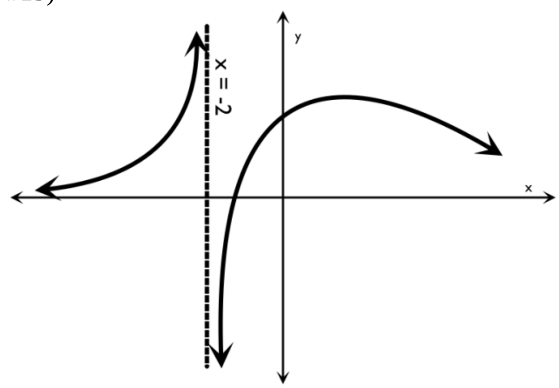


a. $\lim_{x \to 0^-} f(x)$

b. $\lim_{x \to 0^+} f(x)$

c. $\lim_{x \to 0} f(x)$

#13)

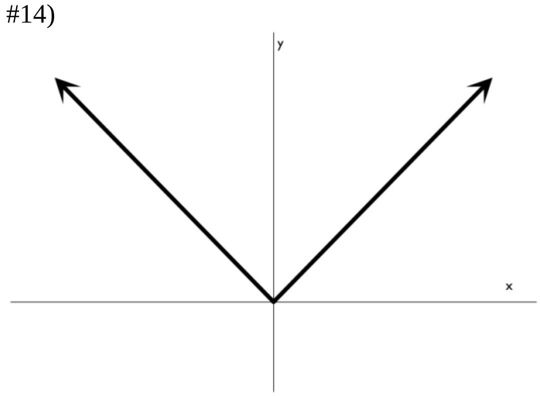


a. $\lim_{x \to -2^-} f(x)$

b. $\lim_{x \to -2^+} f(x)$

c. $\lim_{x \to -2} f(x)$

#14)



a. $\lim_{x \to 0^-} f(x)$

b. $\lim_{x \to 0^+} f(x)$

c. $\lim_{x \to 0} f(x)$

D: Draw each graph by hand. Find the limit as x approaches 3 from the left and from the right. Find the two sided limit.

#15) $f(x) = \begin{cases} x & if\ x < 3 \\ x - 6 & if\ x \geq 3 \end{cases}$

#16) $f(x) = \begin{cases} 2x + 1 & if\ x < 3 \\ -2x - 1 & if\ x \geq 3 \end{cases}$

#17) $f(x) = \begin{cases} \frac{1}{3}x + 27 & if\ x < 3 \\ x - 1 & if\ x \geq 3 \end{cases}$

#18) $f(x) = \begin{cases} 2x & if\ x < 3 \\ -2x + 12 & if\ x \geq 3 \end{cases}$