Limits & Continuity
1.3 – One Sided Limits & Graphing

One-Sided Limits

In a one-sided limit, the variable approaches the number from one side only. For example, the limit as x approaches 3 from the left, denoted $x \to 3^-$, means the limit using only x-values to the left of 3, such as 2.9, 2.99, 2.999…. The limit as x approaches 3 from the right, denoted $x \to 3^+$, means the limit using only x-values to the right of 3, such as 3.1, 3.01, 3.001….

Left and Right Limits

$\lim_{x \to c^-} f(x)$ means the limit of *f(x)* as $x \to c$, but with $x < c$.

$\lim_{x \to c^+} f(x)$ means the limit of *f(x)* as $x \to c$, but with $x > c$.

Two-sided, Left and Right Limits

$\lim_{x \to c} f(x) = L$ iff both one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and equal the same number L.

Ex A: Translate into words:

$x \to 3^-$

$x \to -3$

$x \to -3^-$

Ex B: For the piecewise linear function $f(x) = \begin{cases} x+1 & if\ x \le 3 \\ 8-2x & if\ x > 3 \end{cases}$ find the following limits by direct substitution.

a. $\lim_{x \to 3^-} f(x)$

$\lim_{x \to 3^-} (x+1) = (3)+1$
$= 4$

b. $\lim_{x \to 3^+} f(x)$

$\lim_{x \to 3^+} (8-2x) = 8-2(3)$
$= 8-6$
$= 2$

c. $\lim_{x \to 3} f(x) =$ dne

Limits of Functions of Two Variables

Some limits involve two variables, with only one variable approaching a limit.

Ex C: Finding a Limit of a Function of Two Variables

#1) Find $\lim_{h \to 0} (x^2 + xh + h^2)$

$= x^2 + x(0) + (0)^2$
$= x^2$

#2) Find $\lim_{h \to 0} (3x^2 + 5xh + 1)$

$= 3x^2 + 5x(0) + 1$
$= 3x^2 + 1$

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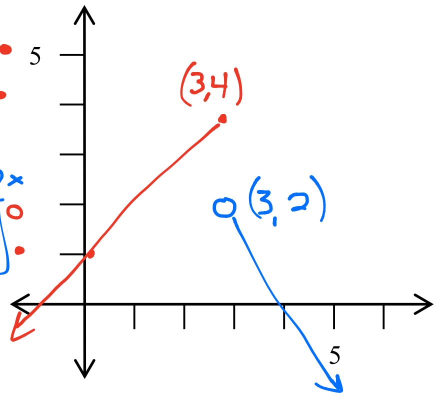
Ex D: Finding One-Sided Limits

For the piecewise linear function $f(x) = \begin{cases} x+1 & if\ x \le 3 \\ 8-2x & if\ x > 3 \end{cases}$ find the following limits by graphing.

x	x+1
3	4
0	1

x	8-2x
3	2
5	-2



a. $\lim\limits_{x\to 3^-} f(x)$ $=4$

b. $\lim\limits_{x\to 3^+} f(x)$ $=2$

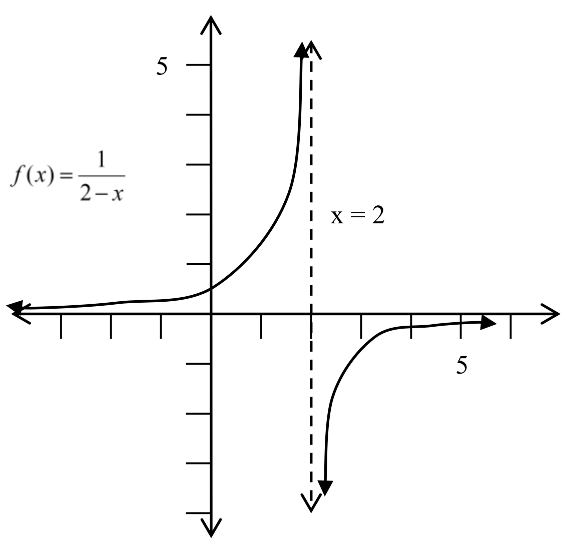
c. $\lim\limits_{x\to 3} f(x)$ $=$ d.n.e.

Infinite Limits

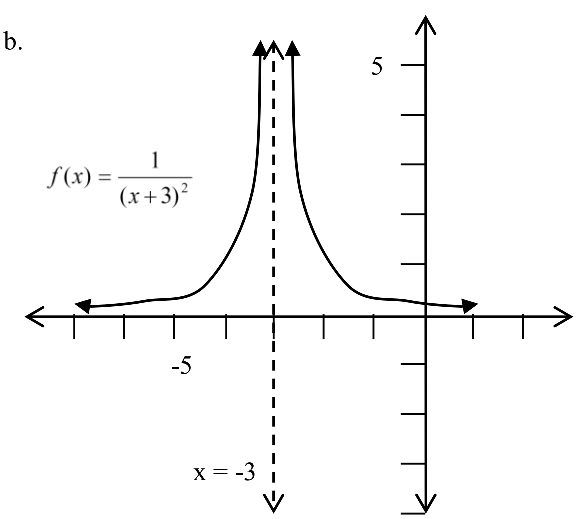
We may use the symbols ∞ (infinity) and -∞ (negative infinity) to indicate that the values of a function become arbitrarily large positive or arbitrarily large negative. Dashed lines on a graph, where a function approaches ∞ or -∞, are called vertical asymptotes.

Ex E: Finding Limits Involving ±∞.
For each function graphed below, use the limit notation with ∞ and -∞ to describe its behavior as *x* approaches the vertical asymptote from the left, from the right, and from both sides.

a.



b.



a.

From the left $\lim\limits_{x\to 2^-} f(x) = \infty$, dne

From the right $\lim\limits_{x\to 2^+} f(x) = -\infty$, dne

From both sides $\lim\limits_{x\to 2} f(x) =$ dne

b.

From the left $\lim\limits_{x\to 3^-} f(x) = \infty$, dne

From the right $\lim\limits_{x\to 3^+} f(x) = \infty$, dne

From the right $\lim\limits_{x\to 3} f(x) = \infty$, dne

Careful. To say that a limit exists is to say that it is a single number. Since ∞ is not a number, if $\lim\limits_{x\to c} f(x) = \infty$, then the limit does not exists (d.n.e.)

A: Translate into words:

#1)

$x \to 10^-$ x approaches 10 from the left.

$x \to -10$ x approaches negative 10 from both sids.

$x \to -10^-$ x approaches negative 10 from the left.

B: Find each limit by substitution.

#2) $f(x) = \begin{cases} x+6 & if\ x \le 2 \\ 2x-5 & if\ x > 2 \end{cases}$

$\lim_{x\to 2^-} (x+6) = (2)+6 = 8$

$\lim_{x\to 2^+} (2x-5) = 2(2)-5 = 4-5 = -1$

a. $\lim_{x\to 2^-} f(x)$ $= 8$

b. $\lim_{x\to 2^+} f(x)$ $= -1$

c. $\lim_{x\to 2} f(x)$ $=$ dne

#3) $f(x) = \begin{cases} -3x+6 & if\ x \le 0 \\ 2x+6 & if\ x > 0 \end{cases}$

$\lim_{x\to 0^-} (-3x+6) = -3(0)+6 = 6$

$\lim_{x\to 0^-} (2x+6) = 2(0)+6 = 6$

a. $\lim_{x\to 0^-} f(x)$ $= 6$

b. $\lim_{x\to 0^+} f(x)$ $= 6$

c. $\lim_{x\to 0} f(x)$ $= 6$

B: Find each limit

#4) $f(x) = |x|$

$f(x) = \begin{cases} x & if\ x \ge 0 \\ -x & if\ x < 0 \end{cases}$

a. $\lim_{x\to 0^-} f(x)$ $= \lim_{x\to 0^-} -x = -(0) = 0$

b. $\lim_{x\to 0^+} f(x)$ $= \lim_{x\to 0^+} x = (0) = 0$

c. $\lim_{x\to 0} f(x)$ $= 0$

#5) $f(x) = \frac{|x|}{x}$

If $x<0$, then
$f(x) = \frac{|-x|}{-x}$
$= \frac{x}{-x}$
$f(x) = -1$
$\therefore f(x) = -1$ if $x<0$

If $x>0$, then
$f(x) = \frac{|+x|}{+x}$
$= \frac{+x}{+x}$
$f(x) = 1$
$\therefore f(x) = 1$ if $x>0$

$f(x) = \begin{cases} -1 & if\ x<0 \\ 1 & if\ x>0 \end{cases}$

a. $\lim_{x\to 0^-} f(x)$ $= -1$

b. $\lim_{x\to 0^+} f(x)$ $= 1$

c. $\lim_{x\to 0} f(x)$ $=$ dne

#6) $f(x) = \frac{-x}{|x|}$

If $x<0$, then
$f(x) = \frac{-(-x)}{|-x|}$
$= \frac{x}{x}$
$f(x) = 1$
$\therefore f(x) = 1$ if $x<0$

If $x>0$, then
$f(x) = \frac{-(+x)}{|+x|}$
$= \frac{-x}{x}$
$f(x) = -1$
$\therefore f(x) = -1$ if $x>0$

$f(x) = \begin{cases} 1 & if\ x<0 \\ -1 & if\ x>0 \end{cases}$

a. $\lim_{x\to 0^-} f(x)$ $= 1$

b. $\lim_{x\to 0^+} f(x)$ $= -1$

c. $\lim_{x\to 0} f(x)$ $=$ dne

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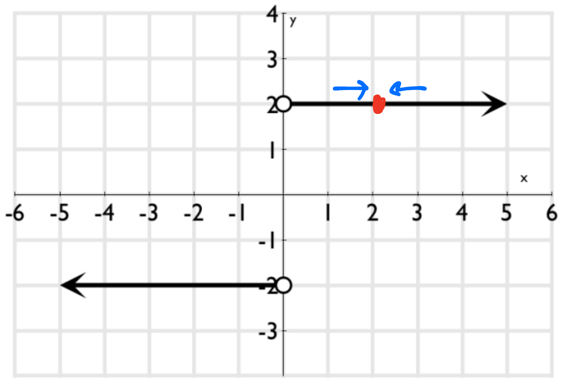
C: Find each limit.

#7)

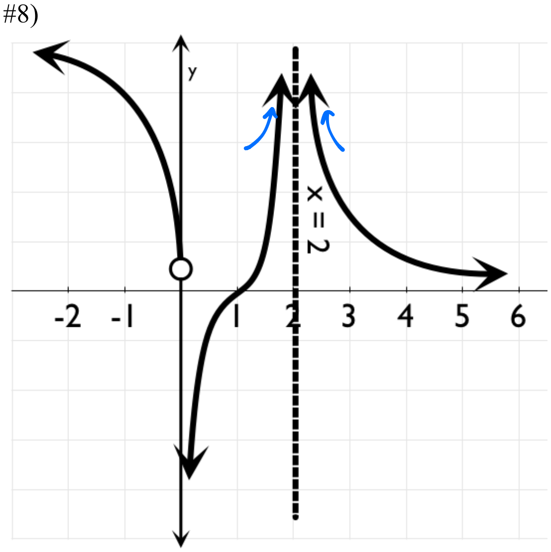


a. $\lim_{x \to 2^-} f(x)$ = 2

b. $\lim_{x \to 2^+} f(x)$ = 2

c. $\lim_{x \to 2} f(x)$ = 2

#8)

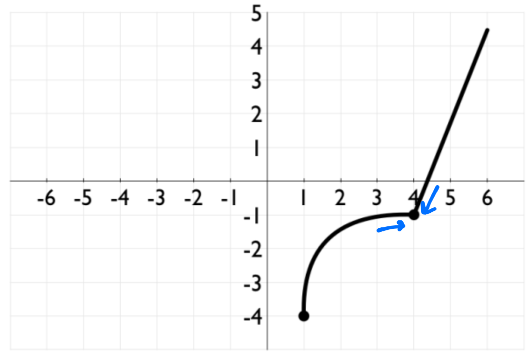


a. $\lim_{x \to 2^-} f(x)$ = ∞, dne

b. $\lim_{x \to 2^+} f(x)$ = ∞, dne

c. $\lim_{x \to 2} f(x)$ = ∞, dne

#9)

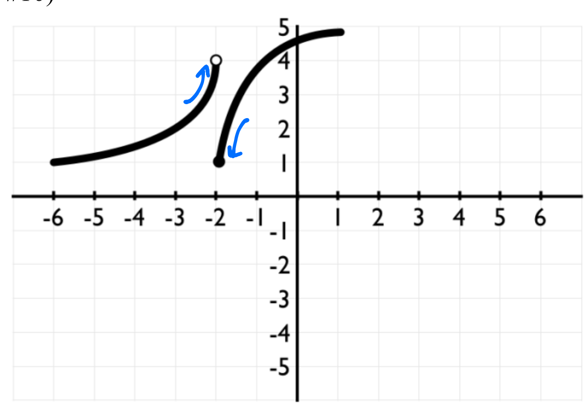


a. $\lim_{x \to 4^-} f(x)$ = -1

b. $\lim_{x \to 4^+} f(x)$ = -1

c. $\lim_{x \to 4} f(x)$ = -1

#10)



a. $\lim_{x \to -2^-} f(x)$ = 4

b. $\lim_{x \to -2^+} f(x)$ = 1

c. $\lim_{x \to -2} f(x)$ = dne

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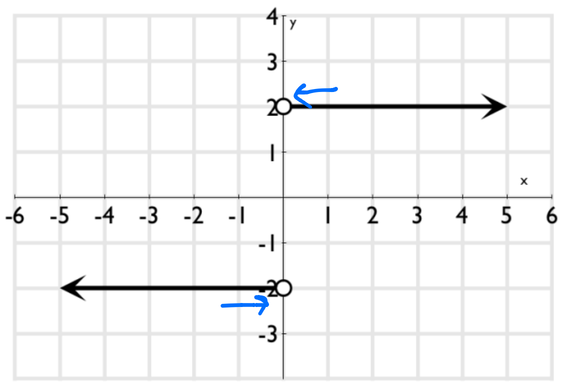
C: Find each limit.

#11)

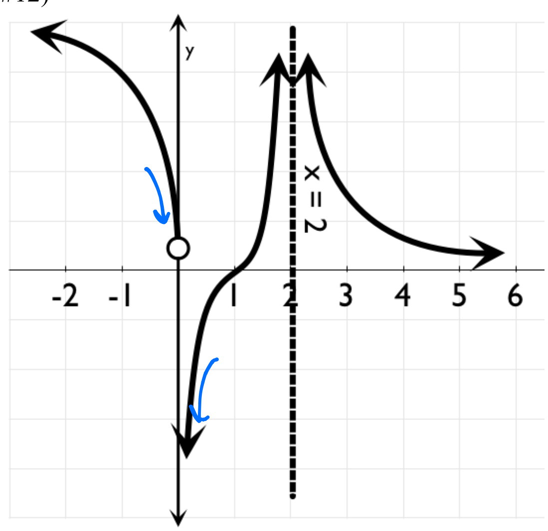


a. $\lim_{x\to 0^-} f(x)$ = -2

b. $\lim_{x\to 0^+} f(x)$ = 2

c. $\lim_{x\to 0} f(x)$ = dne

#12)

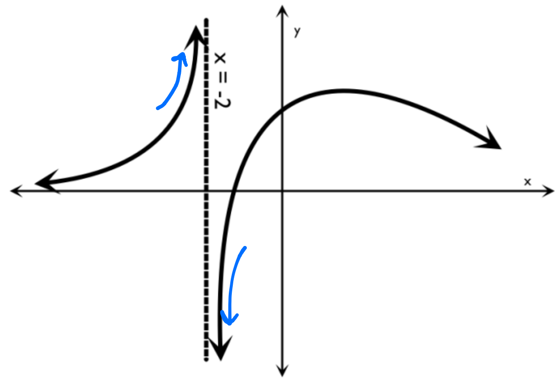


a. $\lim_{x\to 0^-} f(x)$ = $\frac{1}{2}$

b. $\lim_{x\to 0^+} f(x)$ = $-\infty$, dne

c. $\lim_{x\to 0} f(x)$ = dne

#13)

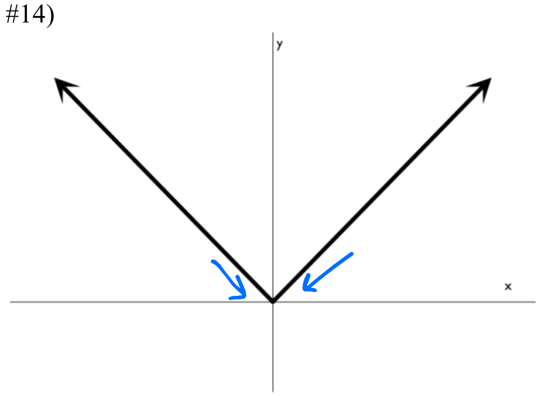


a. $\lim_{x\to -2^-} f(x)$ = ∞, dne

b. $\lim_{x\to -2^+} f(x)$ = $-\infty$, dne

c. $\lim_{x\to -2} f(x)$ = dne

#14)



a. $\lim_{x\to 0^-} f(x)$ = 0

b. $\lim_{x\to 0^+} f(x)$ = 0

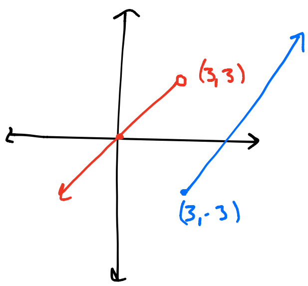
c. $\lim_{x\to 0} f(x)$ = 0

D: Draw each graph by hand. Find the limit as x approaches 3 from the left and from the right. Find the two sided limit.

#15) $f(x) = \begin{cases} x & if\ x < 3 \\ x - 6 & if\ x \geq 3 \end{cases}$

x	x
3	3
0	0

x	x − 6
3	−3
6	0



$\lim_{x \to 3^-} f(x) = 3$

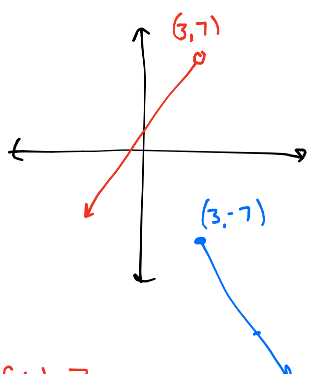
$\lim_{x \to 3^+} f(x) = -3$

$\lim_{x \to 3} f(x) =$ dne

#16) $f(x) = \begin{cases} 2x + 1 & if\ x < 3 \\ -2x - 1 & if\ x \geq 3 \end{cases}$

x	2x+1
3	7
0	1

x	−2x−1
3	−7
6	−13



$\lim_{x \to 3^-} f(x) = 7$

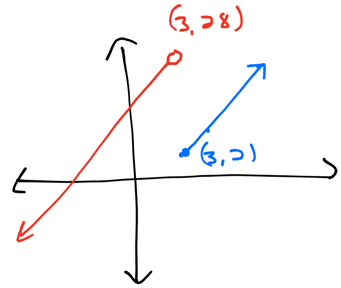
$\lim_{x \to 3^+} f(x) = -7$

$\lim_{x \to 3} f(x) =$ dne

#17) $f(x) = \begin{cases} \frac{1}{3}x + 27 & if\ x < 3 \\ x - 1 & if\ x \geq 3 \end{cases}$

x	$\frac{1}{3}x+27$
3	28
0	27

x	x − 1
3	2
6	5



$\lim_{x \to 3^-} f(x) = 28$

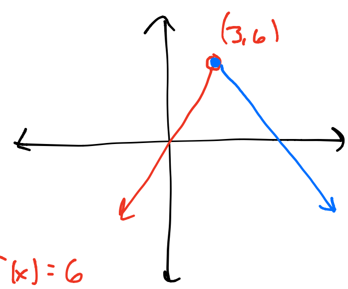
$\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x) =$ dne

#18) $f(x) = \begin{cases} 2x & if\ x < 3 \\ -2x + 12 & if\ x \geq 3 \end{cases}$

x	2x
3	6
0	0

x	−2x+12
3	6
6	0



$\lim_{x \to 3^-} f(x) = 6$

$\lim_{x \to 3^+} f(x) = 6$

$\lim_{x \to 3} f(x) = 6$